MATHEMATICS 151

Assignment 2, due Friday 02 July 1999

Section 1.1 (pg. 50):

2. $y = 1 + x + x^2$.
$P = (1, 3)$, $Q = (x, 1 + x + x^2)$.
$$m_{PQ} = \frac{(1 + x + x^2) - 3}{x - 1} = \frac{x^2 + x - 2}{x - 1} = x + 2, \quad x \neq 1.$$

(a)

x	2.0	1.5	1.1	1.01	1.001
m_{PQ}	4.0	3.5	3.1	3.01	3.001

x	0.0	0.5	0.9	0.99	0.999
m_{PQ}	2.0	2.5	2.9	2.99	2.999

(b)   The slope m of the tangent line at $P = (1, 3)$ appears to be 3.

(c)   The tangent line at $P = (1, 3)$ has equation $y - 3 = 3(x - 1)$, or $y = 3x$.
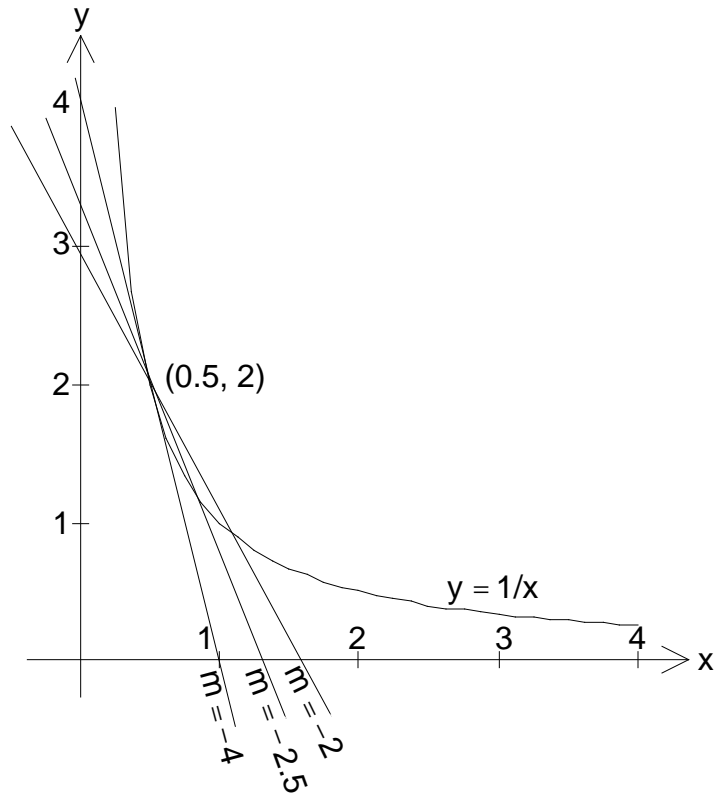
4. $P = (0.5, 2)$, $Q = (x, 1/x)$.
$$m_{PQ} = \frac{\frac{1}{x} - 2}{x - 0.5} = \frac{1 - 2x}{x(x - 0.5)} = -\frac{2}{x},$$
$x \neq 0$ and $x \neq 0.5$.

(a)   See table of Secant Line Slopes below.

(b)   The slope m of the tangent line at $P = (0.5, 2)$ appears to be -4.

(c)   The tangent line at $P = (0.5, 2)$ has equation $y - 2 = -4(x - 0.5)$, or $y = -4x + 4$.

(d)   See graph above and to the right.

For Exercise 4

Secant Line Slopes

x	2.0	1.0	0.9	0.8	0.7
m_{PQ}	-1.0	-2.0	-2.222222	-2.5	-2.857143

x	0.6	0.55	0.51	0.45	0.49
m_{PQ}	-3.333333	-3.636364	-3.921569	-4.444444	-4.081633

6. $h = 58t - 0.83t^2$

(a) Note when $t = 1$, $h = 57.17$. The average velocity over the time interval from $t = 1$ to another value of t is $\dfrac{58t - 0.83t^2 - 57.17}{t - 1} = -0.83t + 57.17$, $t \neq 1$.

t	2.0	1.5	1.1	1.01	1.001
Average velocity from 1 to t	55.51	55.925	56.257	56.3317	56.33917

In each case t is measured in seconds and the average velocity is measured in metres per second.

(b) The instantaneous velocity at $t = 1$ would appear to be 56.34 m/s.

8. The position of a car is as indicated in the table.

| t (seconds)	0	1	2	3	4	5
s (feet)	0	10	32	70	119	178

(a) Table of average velocities beginning at $t = 2$ and lasting for the indicated duration in seconds

Duration	Average Velocity
3	48.666667
2	43.500000
1	38.000000

(b) See graph to the right.
The given points were plotted and a
smooth curve was drawn through them.
A tangent line at (2, 32) was then drawn.
Its slope appears to be about 30.
The instantaneous velocity when $t = 2$ is about 30 ft/s.

s

(5, 178)

(4, 119)

(3, 70)

(2, 32)

(1, 10)

(0, 0)

t
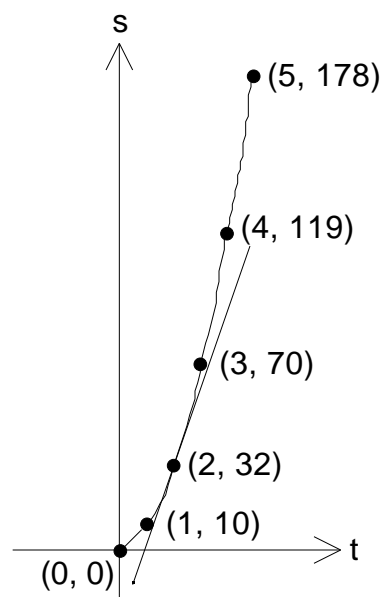
For Exercise 8

Section 1.2 (pg. 59):

4. (a) $\lim_{x \to 1} g(x) = 0$.

(b) $\lim_{x \to 0} g(x)$ does not exist. If x is close to 0 then $g(x)$ is numerically very large, but of opposite sign to x.

(c) $\lim_{x \to 2} g(x) = 1$, not 1.5. The fact that $g(2) = 1.5$ is irrelevant.

(d) $\lim_{x \to -2} g(x) = 0$. The "corner" at $(-2, 0)$ is harmless for the limit of $g(x)$.

(e) $\lim_{x \to -1^-} g(x) = 1$. The fact that $g(-1) = 0$ is irrelevant.

(f) $\lim_{x \to -1} g(x)$ does not exist. If x is close to -1 and $x < -1$ then $g(x)$ is close to 1, but if x is close to -1 and $x > -1$ then $g(x)$ is close to 0, not 1.
The fact that $g(-1) = 0$ is irrelevant.

10. See the table of values for the function

$g(x) = \dfrac{1 - x^2}{x^2 + 3x - 10}$, $x \ne -5$, $x \ne 2$, to the right.

From the table it would appear that

$\lim\limits_{x \to 2^+} \dfrac{1 - x^2}{x^2 + 3x - 10} = -\infty$.

In fact the numerator, $1 - x^2$, gets close to -3 and the denominator, $x^2 + 3x - 10$, gets close to 0 but stays positive when x gets close to 2 but stays to its right, so the limit **is** $-\infty$.

x	g(x)
3.0	-1.000000
2.1	-4.802817
2.01	-43.368046
2.001	-429.081703
2.0001	-4286.224497
2.00001	-42857.714287

For Exercise 10

14. See the table of values for the function

$g(x) = \dfrac{\cos x - 1}{\sin x}$ to the right.

From the table it would appear that $\lim\limits_{x \to 0} g(x) = 0$.

In fact $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} \dfrac{\cos x - 1}{\sin x} = \lim\limits_{x \to 0} \dfrac{\cos^2 x - 1}{\sin x(\cos x + 1)} =$

$= \lim\limits_{x \to 0} \dfrac{-\sin^2 x}{\sin x(\cos x + 1)} = \lim\limits_{x \to 0} \dfrac{-\sin x}{\cos x + 1} = 0$

since the numerator, $-\sin x$, gets close to 0 and the denominator, $\cos x + 1$, gets close to 2 when x gets close to 0.

x	g(x)
1.0	-0.546302
0.5	-0.255342
0.4	-0.202710
0.3	-0.151135
0.2	-0.100335
0.1	-0.050042
0.05	-0.025005
0.01	-0.005000

For Exercise 14

16. $\lim\limits_{x \to 5^-} \dfrac{6}{x - 5} = -\infty$ since the denominator, $x - 5$, gets close to 0 but stays to its left when x gets close to 5 but stays to its left, and meanwhile the numerator is always the positive number 6.

24. See the table of values for the function

$M(x) = \dfrac{2^x - 1}{x}$ to the right.

It appears that $\lim\limits_{x \to 0} M(x) = \lim\limits_{x \to 0} \dfrac{2^x - 1}{x} \approx 0.693$.

We will later see that the value of this limit is $\ln 2 \approx 0.693147$. ($\ln$ is the natural logarithm function.)

x	M(x)
0.5	0.828427
0.1	0.717735
0.05	0.705298
0.01	0.695555
0.005	0.694350
0.001	0.693387
0.0005	0.693267
0.0001	0.693171

For Exercise 24

26. $h(x) = \dfrac{\tan x - x}{x^3}$, $x \ne 0$, x not an odd multiple of $\pi/2$.

(a) See table to the right.

(b) From this table, perhaps $\displaystyle\lim_{x \to 0} \dfrac{\tan x - x}{x^3} = \dfrac{1}{3}$.

(c) See table below and to the right.

The values you choose for x in your table might be different from mine. I used my TI–36 calculator.

Is $\displaystyle\lim_{x \to 0} \dfrac{\tan x - x}{x^3} = 0$? The table to the right might lead us to suspect that it is. But the answer is **NO**!

This limit **IS** $\dfrac{1}{3}$ after all!

I set up *Maple* V, Version 3, running on a Macintosh Quadra 700 to use 30 significant figures (default is 10) and asked it to find $h(x)$ for the values of x in this table 0.0003 and smaller.

Here are the results I obtained.

x	h(x)
1.0	0.557407725
0.5	0.370419919
0.1	0.334672085
0.05	0.333667004
0.01	0.333346700
0.005	0.333336640

For Exercise 26 (a)

x	h(x)
0.001	0.333330000
0.0005	0.333336000
0.0004	0.333328125
0.0003	0.333333333
0.0002	0.333375000
0.0001	0.333000000
0.00005	0.333600000
0.00001	0.000000000
0.000001	0.000000000
0.0000001	0.000000000

For Exercise 26(c)

h(0.0003)   0.33333334533333770476222222222
h(0.0002)   0..33333333866666667530158750000000
h(0.0001)   0.33333333466666667206300000000
h(0.00005)   0.33333333366666666700400000000
h(0.00001)   0.33333333334666666670000000000
h(0.000001)   0.33333333333346667000000000000
h(0.0000001)   0.33333333333333350000000000000000

Maple is doing pretty well. But let's push it a bit harder!

h(0.00000001)   0.33333333333333000000000000000000
h(0.000000001)   0.33333333333300000000000000000000
h(0.0000000001)   0.33333333330000000000000000000000
h(0.00000000001)   0.33333333000000000000000000000000
h(0.000000000001)   0.33333000000000000000000000000000
h(0.0000000000001)   0.33300000000000000000000000000000
h(0.00000000000001)   0.30000000000000000000000000000000
h(0.000000000000001)   0.00000000000000000000000000000000

Of course I could have asked for more than 30 significant figures. This would only have postponed the inevitable. All such calculations are eventually doomed to failure!

Read Appendix G, "Lies My Calculator and Computer Told Me" in the text. Later, see L'Hospital's Rule in Section 3.8, which can be used to show that the limit really is $\dfrac{1}{3}$. (The author mistakenly refers to Section 6.8, which does not exist.)

Section 1.3 (pg. 68):

2. $\lim\limits_{x \to -3} (x^3 + 2x^2 + 6) = \lim\limits_{x \to -3} x^3 + \lim\limits_{x \to -3} 2x^2 + \lim\limits_{x \to -3} 6$ (sum property)

$= (\lim\limits_{x \to -3} x)^3 + (\lim\limits_{x \to -3} 2)(\lim\limits_{x \to -3} x)^2 + \lim\limits_{x \to -3} 6$ (product property)

$= (-3)^3 + (2)(-3)^2 + 6 = -3.$ (behaviour of $\lim\limits_{x \to a} c$ and $\lim\limits_{x \to a} x$)

8. $\lim\limits_{x \to 64} \left(\sqrt[3]{x} + 3\sqrt{x}\right) = \left(\sqrt[3]{64} + 3\sqrt{64}\right) = 4 + 3{\cdot}8 = 28.$ (sum, product, root properties)

16. $\lim\limits_{x \to -3} \dfrac{x^2 - x - 12}{x + 3} = \lim\limits_{x \to -3} \dfrac{(x + 3)(x - 4)}{x + 3} = \lim\limits_{x \to -3} (x - 4) = -7.$

52. $\lim\limits_{x \to -2} [\![x]\!]$ does not exist since for $-3 < x < 2$, $[\![x]\!] = -3$ making $\lim\limits_{x \to -2^-} [\![x]\!] = -3$,
while for $-2 < x < -1$, $[\![x]\!] = -2$ making $\lim\limits_{x \to -2^+} [\![x]\!] = -2$.

56. $\lim\limits_{x \to -2^-} \sqrt{x^2 + x - 2} = 0$ since $\sqrt{x^2 + x - 2}$ is defined in $(-\infty, -2]$ $[1, +\infty)$, and its
values are near 0 for $x < -2$ but x near -2. However the right-hand limit at -2
fails to exist because this function is
undefined for $x > -2$ but x near -2.

60. $f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x < 1 \\ 3 - x & \text{if } x \ge 1 \end{cases}$

(a) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 - 2x + 2) = 1,$
and $\lim\limits_{x \to 1^+} (x) = \lim\limits_{x \to 1^+} (3 - x) = 2.$

(b) $\lim\limits_{x \to 1} f(x)$ does not exist since
$\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$ are not the
same.

(c)   See graph to the right.

$f(x) = \begin{cases} x^2 - 2x + 2 & \text{if } x < 1 \\ 3 - x & \text{if } x \ge 1 \end{cases}$

$(1, 2)$ is there

$(0, 2)$

$(1, 1)$ is not there

$(3, 0)$
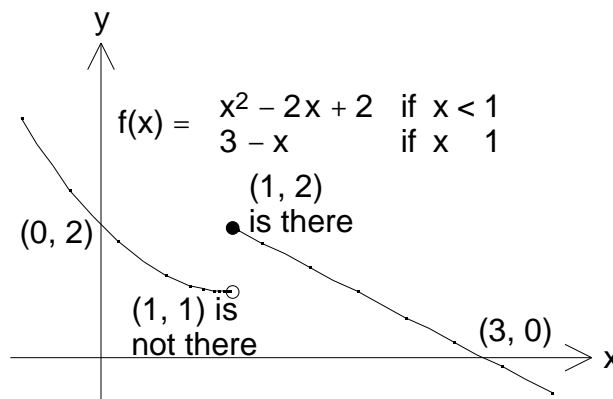
For Exercise 60

70.   Many examples exist. Let $f(x)$ be as in Exercise 60, $g(x) = -f(x)$, $a = 1$.
Then $\lim\limits_{x \to 1} g(x)$ doesn't exist either, but $\lim\limits_{x \to 1} (f(x) + g(x)) = \lim\limits_{x \to 1} 0 = 0.$

Section 1.4 (pg. 78):

2. $|(8x - 5) - 11| = |8x - 16| = 8|x - 2|.$
So $(8x - 5)$ is 8 times as far from 11 as
x is from 2. Consequently we should
be sure that x is at least 8 times closer
to 2 than we want $(8x - 5)$ to be to 11.
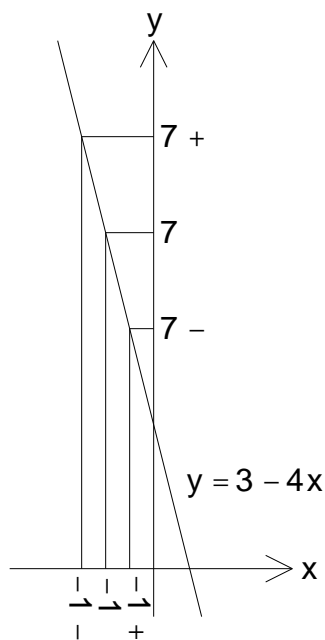
   (a)   0.00125

   (b)   0.000125

   (c)   0.0000125

14. $|(3 - 4x) - 7| = |-4 - 4x| = 4|x - (-1)|.$
So $(3 - 4x)$ is 4 times as far from 7
as x is from -1.
If we want to insure that $|(3 - 4x) - 7| <$
(can be any positive real number we like),

it is enough to take $0 < |x - (-1)| <$, where $= \frac{}{4}$.

A smaller value of than $\frac{}{4}$ will also work.   For Page 78, Exercise 14

(Someone might object that we were being wasteful then, but that's permitted.)
See graph above and to the right.

36.   If $M > 0$ then $\frac{1}{(x + 3)^4} > M$ is equivalent to $\frac{1}{|x - (-3)|^4} > M$, or to $0 < |x - (-3)| < \frac{1}{\sqrt[4]{M}}$.

Thus if $0 < |x - (-3)| <$ where $\frac{1}{\sqrt[4]{M}}$, then $\frac{1}{|x - (-3)|^4} > M$, and according to

Definition 6, $\lim\limits_{x \to -3} \frac{1}{(x + 3)^4} = $.