MATHEMATICS 151

Assignment 3, due Monday 05 July 1999

Section 1.5 (pg. 87):

6. $\lim\limits_{x \to 4} g(x) = \lim\limits_{x \to 4} \dfrac{x+1}{2x^2-1} = \dfrac{\lim\limits_{x \to 4} x + \lim\limits_{x \to 4} 1}{2(\lim\limits_{x \to 4} x)^2 - \lim\limits_{x \to 4} 1} = \dfrac{4+1}{2 \cdot 4^2 - 1} = \dfrac{5}{31} = g(4)$,

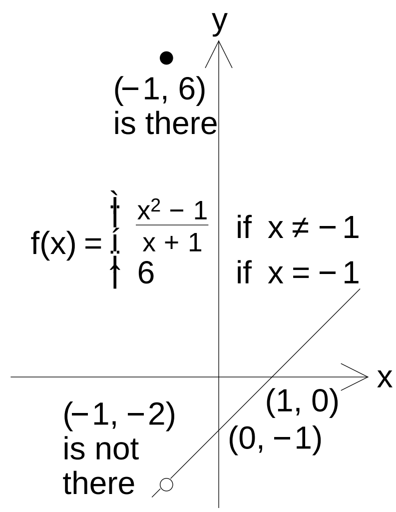
so $g(x) = \dfrac{x+1}{2x^2-1}$ is continuous at $x = 4$.

10. If a $(-\infty, 3)$, $\lim\limits_{x \to a} \dfrac{x+1}{x-3} = \dfrac{a+1}{a-3} = F(a)$, so

$F(x) = \dfrac{x+1}{x-3}$ is continuous at $x = a$ for all a $(-\infty, 3)$.
Thus F is continuous in $(-\infty, 3)$.

14. $f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases} = \begin{cases} x-1 & \text{if } x \neq -1 \\ 6 & \text{if } x = -1 \end{cases}$

$\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} (x-1) = (-1) - 1 = -2$, but $f(-1) = 6$.
So f is discontinuous at -1.

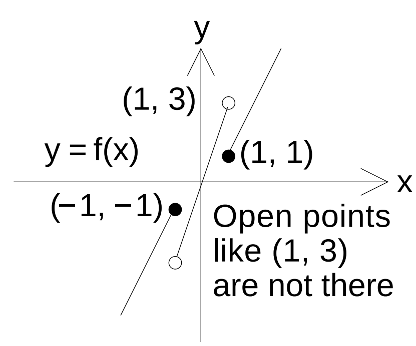


For Exercise 14

22. $f(t) = 2t + \sqrt{25 - t^2}$ has domain $[-5, 5]$. The polynomial functions $g(t) = 2t$ and $h(t) = 25 - t^2$ are continuous everywhere, while the root function $k(t) = \sqrt{t}$ is continuous on $[0, +\infty)$. Thus the composition $(k \circ h)(t) = \sqrt{25 - t^2}$ is continuous on its domain $[-5, 5]$, and the sum $f = g + k \circ h$ is continuous on its domain $[-5, 5]$.

32. $f(x) = \begin{cases} 2x+1 & \text{if } -\infty < x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x-1 & \text{if } 1 \leq x < +\infty \end{cases}$

f is continuous on $(-\infty, -1)$ $(-1, 1)$ $(1, +\infty)$, since on each of these open intervals it is a polynomial.
$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (2x+1) = -1 = f(-1)$.
$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} 3x = -3 \neq f(-1)$.
$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 3x = 3 \neq f(1)$.
$\lim\limits_{x \to 1^+} (x) = \lim\limits_{x \to 1^+} (2x-1) = 1 = f(1)$.



For Exercise 32

So f is discontinuous at -1 and at 1, but continuous from the left at -1 and continuous from the right at 1.

40. (a) $f(x) = \frac{x^2 - 2x - 8}{x + 2} = \frac{(x+2)(x-4)}{(x+2)} = x - 4,\ x \neq -2$. Then $\lim_{x \to -2} f(x) = -6$. Define $g(x) = x - 4,\ x \in (-\infty, +\infty)$. Then g is continuous on $(-\infty, +\infty)$ and agrees with f except at -2, so f has a removable discontinuity at -2.

(b) $f(x) = \frac{x-7}{|x-7|} = \begin{cases} -1 & \text{if } x < 7 \\ 1 & \text{if } x > 7 \end{cases}$ $\lim_{x \to 7} f(x)$ does not exist, so f does not have a removable discontinuity at 7. It has a jump discontinuity there.

(c) $f(x) = \frac{x^3 + 64}{x + 4} = \frac{(x+4)(x^2 - 4x + 16)}{x + 4} = x^2 - 4x + 16,\ x \neq -4$. Then $\lim_{x \to -4} f(x) = 48$. Define $g(x) = x^2 - 4x + 16,\ x \in (-\infty, +\infty)$. Then g is continuous on $(-\infty, +\infty)$ and agrees with f except at -4, so f has a removable discontinuity at -4.

(d) $f(x) = \frac{3 - \sqrt{x}}{9 - x} = \frac{\left(3 - \sqrt{x}\right)\left(3 + \sqrt{x}\right)}{(9-x)\left(3 + \sqrt{x}\right)} = \frac{9 - x}{(9-x)\left(3 + \sqrt{x}\right)} = \frac{1}{3 + \sqrt{x}}$, if $0 \leq x < 9$ or $9 < x < +\infty$, so $\lim_{x \to 9} f(x) = \frac{1}{6}$. Now define $g(x) = \frac{1}{3 + \sqrt{x}},\ x \in [0, +\infty)$. Then g is continuous on $[0, +\infty)$ and agrees with f except at 9, so f has a removable discontinuity at 9.

44. If $g(x) = x^5 - 2x^3 + x^2 + 2$, then $g(-2) = -10$ and $g(-1) = 4$ so by the Intermediate Value Theorem there is a number $c \in (-2, -1)$ with $g(c) = -1$. In fact, $c \approx -1.723594071$. (See Section 2.10.)

Section 1.6 (pg. 100):

2. $\lim_{x \to +\infty} \frac{5 + 2x}{3 - x} = \lim_{x \to +\infty} \frac{\frac{5}{x} + 2}{\frac{3}{x} - 1} = \frac{2}{-1} = -2$ using the quotient, sum, difference, product, and constant properties and the fact that $\lim_{x \to +\infty} \frac{1}{x} = 0$.

4. $\lim_{t \to +\infty} \frac{7t^3 + 4t}{2t^3 - t^2 + 3} = \lim_{t \to +\infty} \frac{7 + \frac{4}{t^2}}{2 - \frac{1}{t} + \frac{3}{t^3}} = \frac{7}{2}$ using the quotient, sum, difference, product, and constant properties and the fact that $\lim_{t \to +\infty} \frac{1}{t} = 0$.

10. $\lim_{t \to -\infty} \frac{6t^2 + 5t}{(1-t)(2t-3)} = \lim_{t \to -\infty} \frac{6 + \frac{5}{t}}{\left(\frac{1}{t} - 1\right)\left(2 - \frac{3}{t}\right)} = -3$.

16. $\lim\limits_{x\to-\infty}\left\{x+\sqrt{x^2+2x}\right\} = \lim\limits_{x\to-\infty}\dfrac{\left(x+\sqrt{x^2+2x}\right)\left(x-\sqrt{x^2+2x}\right)}{x-\sqrt{x^2+2x}} = \lim\limits_{x\to-\infty}\dfrac{-2x}{x-\sqrt{x^2+2x}} =$

$= \lim\limits_{x\to-\infty}\dfrac{-2x}{x-|x|\sqrt{1+\frac{2}{x}}} = \lim\limits_{x\to-\infty}\dfrac{-2}{1+\sqrt{1+\frac{2}{x}}} = -1.$

$\sqrt{x^2} = |x|$, not x, and as soon as $x < 0$, $|x| = -x$; this accounts for the $+$ sign in the denominator in the next to the last step.

20. $\lim\limits_{x\to+\infty}\cos x$ does not exist, since for any $N > 0$ the function $\cos x$ repeatedly takes on all values between -1 and 1 for $x > N$, never settling down and approaching any one value.

38. $\lim\limits_{x\to-\infty}\dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim\limits_{x\to-\infty}\dfrac{x-9}{|x|\sqrt{4+\frac{3}{x}+\frac{2}{x^2}}} = \lim\limits_{x\to-\infty}\dfrac{1-\frac{9}{x}}{-\sqrt{4+\frac{3}{x}+\frac{2}{x^2}}} = -\dfrac{1}{2}$, and

$\lim\limits_{x\to+\infty}\dfrac{x-9}{\sqrt{4x^2+3x+2}} = \lim\limits_{x\to+\infty}\dfrac{x-9}{|x|\sqrt{4+\frac{3}{x}+\frac{2}{x^2}}} = \lim\limits_{x\to+\infty}\dfrac{1-\frac{9}{x}}{\sqrt{4+\frac{3}{x}+\frac{2}{x^2}}} = \dfrac{1}{2}.$

The horizontal asymptotes are $y = -\frac{1}{2}$ (approached as $x \to -\infty$) and $y = \frac{1}{2}$ (approached as $x \to +\infty$).

52. (a) $\dfrac{1}{\sqrt{x}} < 0.0001 \Leftrightarrow \sqrt{x} > 10000 \Leftrightarrow x > 100000000$. (The symbol "$\Leftrightarrow$" means "if and only if" and is correctly used between two statements just when they have the same truth value – in other words, either they're both true or they're both false. So you can check the truth of one of the statements by checking the truth of the other instead.)

(b) Let $\varepsilon > 0$. Then $\dfrac{1}{\sqrt{x}} < \varepsilon \Leftrightarrow \sqrt{x} > \dfrac{1}{\varepsilon} \Leftrightarrow x > \dfrac{1}{\varepsilon^2}$. So take $N = \dfrac{1}{\varepsilon^2}$ in Definition 5.

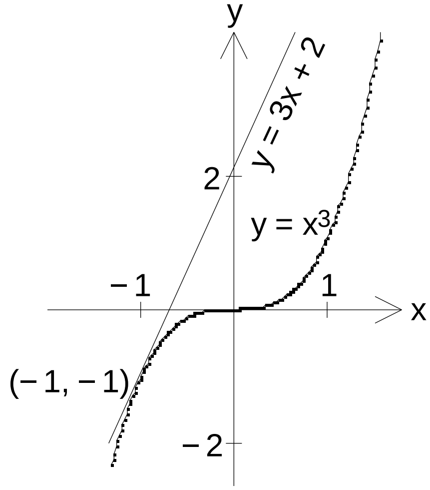
Section 1.7 (pg. 108):

2. $y = x^3$ goes through $(-1, -1)$.

(a) The slope of the tangent line at $(-1, -1)$ is $m = \lim_{x \to -1} \frac{x^3 - (-1)^3}{x - (-1)} = \lim_{x \to -1} \frac{x^3 + 1}{x + 1} =$
$= \lim_{x \to -1} \frac{(x+1)(x^2 - x + 1)}{x + 1} = \lim_{x \to -1} (x^2 - x + 1) = 3.$
Alternatively we can calculate the slope as
$m = \lim_{h \to 0} \frac{(-1+h)^3 - (-1)^3}{h} = \lim_{h \to 0} \frac{3h - 3h^2 + h^3}{h} =$
$= \lim_{h \to 0} \frac{(3 - 3h + h^2)h}{h} = \lim_{h \to 0} (3 - 3h + h^2) = 3.$

(b) The tangent line at $(-1, -1)$ has equation $y - (-1) = 3(x - (-1))$, or $y = 3x + 2$.

(c) See graph to the right.



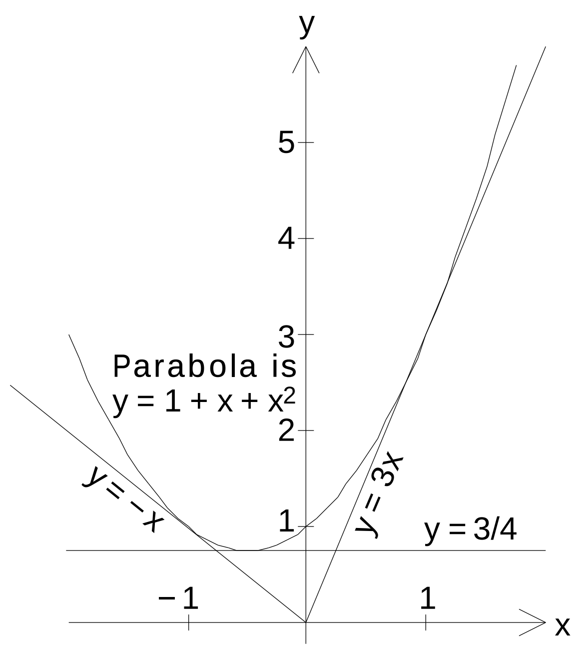
4. $m = \lim_{x \to 1} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1}}}{x - 1} = \lim_{x \to 1} \frac{1 - \sqrt{x}}{\sqrt{x}(x-1)} = \lim_{x \to 1} \frac{1 - \sqrt{x}}{\sqrt{x}\left(\sqrt{x} - 1\right)\left(\sqrt{x} + 1\right)} = \lim_{x \to 1} \frac{-1}{\sqrt{x}\left(\sqrt{x} + 1\right)} = -\frac{1}{2}$
is the slope of the tangent line to $y = \frac{1}{\sqrt{x}}$ at $(1, 1)$. The tangent line has equation
$y - 1 = -\frac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + \frac{3}{2}$.

6. $m = \lim_{x \to 0} \frac{\frac{x}{1-x} - \frac{0}{1-0}}{x - 0} = \lim_{x \to 0} \frac{x}{x(1 - x)} = \lim_{x \to 0} \frac{1}{1 - x} = 1$ is the slope of the tangent line to $y = x/(1 - x)$ at $(0, 0)$. The tangent line has equation $y - 0 = 1(x - 0)$, or $y = x$.

8. (a) $m = \lim_{x \to a} \frac{(1 + x + x^2) - (1 + a + a^2)}{x - a} =$
$= \lim_{x \to a} \frac{(x - a) + (x^2 - a^2)}{x - a} = \lim_{x \to a} \frac{(x - a)(1 + x + a)}{x - a} =$
$= \lim_{x \to a} (1 + x + a) = 1 + 2a$ is the slope of the tangent line to the parabola $y = 1 + x + x^2$ at the point $(a, 1 + a + a^2)$.

(b) The tangent lines at the points $(-1, 1)$, $(-1/2, 3/4)$, and $(1, 3)$ have slopes -1, 0, and 3 and equations $y = -x$, $y = 3/4$, and $y = 3x$ respectively.

(c) See graph to the right.



12. $h = 58t - 0.83t^2$

(a) Note when $t = 1$, $h = 57.17$. The average velocity over the time interval from $t = 1$ to another value of t is $\frac{58t - 0.83t^2 - 57.17}{t - 1} = -0.83t + 57.17$, $t \neq 1$.

t	2.0	1.5	1.1	1.01	1.001
Average velocity from 1 to t	55.51	55.925	56.257	56.3317	56.33917

In each case t is measured in seconds and the average velocity is measured in metres per second. The instantaneous velocity at $t = 1$ would appear to be 56.34 m/s. And of course $\lim_{t \to 1}(-0.83t + 57.17) = 56.34$.

(b) Look at the average velocity v_{av} from time t to time u; let u approach t.

$$v_{av} = \frac{(58u - 0.83u^2) - (58t - 0.83t^2)}{u - t} = \frac{58(u - t) - 0.83(u^2 - t^2)}{u - t} = 58 - 0.83(u + t).$$

As u gets close to t, this gets close to $58 - 1.66t$, so the velocity at time t is $58 - 1.66t$, measured in metres per second when the height is measured in metres and the time is measured in seconds.

(c) The height h is zero if $t = 0$ or if $t = \frac{58}{0.83} \approx 69.88$, so the arrow hits the moon in about 69.88 seconds.

(d) The velocity of the arrow as it hits the moon is $58 - 1.66\left(\frac{58}{0.83}\right) = -58$ m/s. (It's falling then, just as fast as it was rising when it left the bow.)

18. (a) (i) Average rate of growth from 1992 to 1996 is $\frac{164 - 117}{4} = 11.75$.

(ii) Average rate of growth from 1992 to 1995 is $\frac{150 - 117}{3} = 11.0$.

(iii) Average rate of growth from 1992 to 1994 is $\frac{137 - 117}{2} = 10.0$.

(iv) Average rate of growth from 1992 to 1993 is $\frac{126 - 117}{1} = 9.0$.

(b) The instantaneous growth rate at the beginning of 1992 appears to be about 8. One could draw a smooth curve through all these points, sketch a tangent line at the point (1992, 117), and measure its slope. But many different smooth curves pass through these points and have different slopes at (1992, 117).

20. $V(t) = 100000\left(1 - \frac{t}{60}\right)^2$, $0 \leq t \leq 60$, so $V(20) = \frac{400000}{9} = \frac{250}{9} \cdot 1600$,

$V(20 + h) = 100000\left(1 - \frac{20 + h}{60}\right)^2 = 100000 \cdot \left(\frac{40 - h}{60}\right)^2 = \frac{250}{9} \cdot (1600 - 80h + h^2)$,

and $\frac{V(20 + h) - V(20)}{h} = \frac{250}{9} \cdot \left[\frac{(1600 - 80h + h^2) - 1600}{h}\right] = \frac{250}{9} \cdot (h - 80)$.

As h gets close to zero, this gets close to $-\frac{20000}{9}$.

So after 20 minutes, water is flowing out of the tank at a rate of $\frac{20000}{9}$ gal/min.