MATHEMATICS 151

Assignment 4, due Wednesday 07 July 1999

Section 2.1 (pg. 121):

6. $f(t) = 2t^3 - t + 1$; $f(2 + h) = 2(2 + h)^3 - (2 + h) + 1 = 2h^3 + 12h^2 + 23h + 15$ and $f(2) = 15$. Then $\frac{f(2+h) - f(2)}{h} = \frac{(2h^3 + 12h^2 + 23h + 15) - 15}{h} = \frac{2h^3 + 12h^2 + 23h}{h} =$
$= 2h^2 + 12h + 23$, if $h \neq 0$. Here time t is measured in seconds, distance s is measured in metres, and velocity is measured in metres per second. If $g(t)$ is the average velocity from 2 to t, $g(2 + h) = 2h^2 + 12h + 23$, and the instantaneous velocity at $t = 2$ is $\lim_{h \to 0} (2h^2 + 12h + 23) = 23$.

10. $f(x) = \frac{x}{x^2 - 1}$, so $f'(a) = \lim_{x \to a} \frac{\frac{x}{x^2-1} - \frac{a}{a^2-1}}{x - a} = \lim_{x \to a} \frac{(ax + 1)(a - x)}{(x - a)(x^2 - 1)(a^2 - 1)} =$
$= \lim_{x \to a} \frac{-(ax + 1)}{(x^2 - 1)(a^2 - 1)} = -\frac{a^2 + 1}{(a^2 - 1)^2}$.
Alternatively $f'(a) = \lim_{h \to 0} \frac{\frac{a+h}{(a+h)^2-1} - \frac{a}{a^2-1}}{h} = \lim_{h \to 0} \frac{-[a(a + h) + 1]h}{h[(a + h)^2 - 1][a^2 - 1]} =$
$= \lim_{h \to 0} \frac{-[a(a + h) + 1]}{[(a + h)^2 - 1][a^2 - 1]} = -\frac{a^2 + 1}{(a^2 - 1)^2}$.

12. $f(x) = \sqrt{x - 1}$. Note $x - a = [x - 1] - [a - 1] = \left[\sqrt{x-1} - \sqrt{a-1}\right]\left[\sqrt{x-1} + \sqrt{a-1}\right]$.
So $f'(a) = \lim_{x \to a} \frac{\sqrt{x-1} - \sqrt{a-1}}{x - a} = \lim_{x \to a} \frac{1}{\sqrt{x-1} + \sqrt{a-1}} = \frac{1}{2\sqrt{a-1}}$.
Alternatively $h = [a + h - 1] - [a - 1] = \left[\sqrt{a+h-1} - \sqrt{a-1}\right]\left[\sqrt{a+h-1} + \sqrt{a-1}\right]$.
So $f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h-1} - \sqrt{a-1}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{a+h-1} + \sqrt{a-1}} = \frac{1}{2\sqrt{a-1}}$

14. $\lim_{h \to 0} \frac{(2 + h)^3 - 8}{h} = \lim_{h \to 0} \frac{(2 + h)^3 - 2^3}{h} = f'(2)$, where $f(x) = x^3$.

16. $\lim_{x \to 3\pi} \frac{\cos x + 1}{x - 3\pi} = \lim_{x \to 3\pi} \frac{\cos x - \cos(3\pi)}{x - 3\pi} = f'(3\pi)$, where $f(x) = \cos x$.

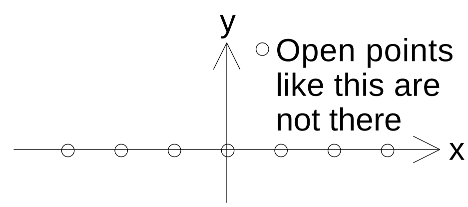
18. $\lim_{x \to 0} \frac{3^x - 1}{x} = \lim_{x \to 0} \frac{3^x - 3^0}{x - 0} = f'(0)$, where $f(x) = 3^x$.

54. If x is between integers, then $f'(x) = \lim_{h \to 0} \frac{[\![x + h]\!] - [\![x]\!]}{h} = \lim_{h \to 0} 0 = 0$ since $[\![x + h]\!]$ and $[\![x]\!]$ are the same when h is small enough. But if x is an integer, $[\![x + h]\!] = x - 1$ if $h < 0$ and h is numerically small enough, $[\![x + h]\!] = x$ if $h > 0$ and h is numerically small enough, and $[\![x]\!] = x$. Thus

$$\lim_{h \to 0^+} \frac{[\![x + h]\!] - [\![x]\!]}{h} = \lim_{h \to 0^+} \frac{x - x}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0, \text{ and}$$

$$\lim_{h \to 0^-} \frac{[\![x + h]\!] - [\![x]\!]}{h} = \lim_{h \to 0^-} \frac{(x - 1) - x}{h} = \lim_{h \to 0^-} \frac{-1}{h}$$

does not exist, so $f'(x)$ does not exist when x is an integer; $f'(x) = 0$ with domain the set of all real numbers that are **not** integers.



For Exercise 54

56. $g(x) = \begin{cases} \frac{x^3 - x}{x^2 + x} & \text{if } x < 1 \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \\ 1 - x & \text{if } x \geq 1 \end{cases}$

$g(-1)$ is undefined.

$$\lim_{x \to -1} \frac{x^3 - x}{x^2 + x} = \lim_{x \to -1} \frac{x(x + 1)(x - 1)}{x(x + 1)} = \lim_{x \to -1} (x - 1) = -2.$$

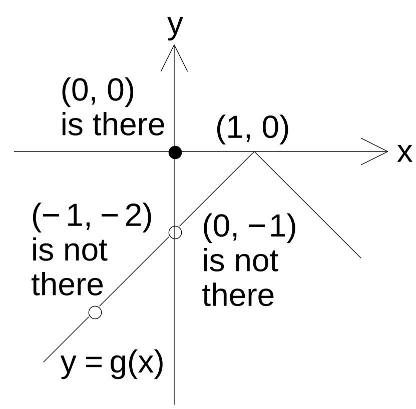
So g has a removable discontinuity at -1.

$$\lim_{x \to 0} \frac{x^3 - x}{x^2 + x} = \lim_{x \to 0} \frac{x(x + 1)(x - 1)}{x(x + 1)} = \lim_{x \to 0} (x - 1) = -1 \neq g(0).$$

So g is discontinuous, hence not differentiable, at 0.

$$\lim_{x \to 1^-} \frac{x^3 - x}{x^2 + x} = \lim_{x \to 1^-} \frac{x(x + 1)(x - 1)}{x(x + 1)} = \lim_{x \to 1^-} (x - 1) = 0.$$

$\lim_{x \to 1^+} (1 - x) = 0$ and $g(1) = 0$ also, so g is continuous at 1.



For Exercise 56

Since $g(x) = x - 1$ on $(-\infty, -1)$ $(-1, 0)$ $(0, 1)$, $g'(x) = 1$ there.
Likewise since $g(x) = 1 - x$ on $(1, +\infty)$, $g'(x) = -1$ there. But what about $g'(1)$?

$$\lim_{h \to 0^-} \frac{g(1 + h) - g(1)}{h} = \lim_{h \to 0^-} \frac{[(1 + h) - 1] - [1 - 1]}{h} = \lim_{h \to 0^-} 1 = 1.$$

On the other hand, $\lim_{h \to 0^+} \frac{g(1 + h) - g(1)}{h} = \lim_{h \to 0^+} \frac{[1 - (1 + h)] - [1 - 1]}{h} = \lim_{h \to 0^+} (-1) = -1.$

So $g'(1)$ does not exist, and g is not differentiable at 1.

g has a "left-hand derivative" $g'_-(1) = \lim_{h \to 0^-} \frac{g(1 + h) - g(1)}{h} = 1$ and a "right-hand derivative" $g'_+(1) = \lim_{h \to 0^+} \frac{g(1 + h) - g(1)}{h} = -1$. See graph above and to the right.

60. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} \left(h \sin \frac{1}{h}\right) = 0 \text{ because } -1 \leq \sin \frac{1}{h} \leq 1,$$

which makes $h \sin \frac{1}{h}$ at least as close to 0 as h itself is.

Section 2.2 (pg. 132):

2. If $g(x) = x^{100} + 50x + 1$, then $g'(x) = 100x^{99} + 50$.

6. If $G(y) = (y^2 + 1)(2y - 7)$ then $G'(y) = 2y(2y - 7) + (y^2 + 1)(2) = 6y^2 - 14y + 2$.
Alternatively, $G(y) = 2y^3 - 7y^2 + 2y - 7$ so $G'(y) = 6y^2 - 14y + 2$.

12. If $f(u) = \frac{1 - u^2}{1 + u^2}$ then $f'(u) = \frac{(1 + u^2)(-2u) - (1 - u^2)(2u)}{(1 + u^2)^2} = -\frac{4u}{(1 + u^2)^2}$.

18. If $y = x^{4/3} - x^{2/3}$ then $y' = \frac{4}{3}x^{1/3} - \frac{2}{3}x^{-1/3}$.

38. $y = x^{5/2}$ so $y' = \frac{5}{2}x^{3/2}$ and $y'(4) = 20$.
The tangent line has equation $y - 32 = 20(x - 4)$, or $y = 20x - 48$.

46. $f(x) = 2x^3 - 3x^2 - 6x + 87$.

So $f'(x) = 6x^2 - 6x - 6 = 6(x^2 - x - 1) = 6\left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$.

The graph has horizontal tangents at $\left(\frac{1 \pm \sqrt{5}}{2}, \frac{167 \mp 5\sqrt{5}}{2}\right)$.

64. $y = (x + 5)(x^2 + 7)(x - 3)$.
$y' = (1)(x^2 + 7)(x - 3) + (x + 5)(2x)(x - 3) + (x + 5)(x^2 + 7)(1) = 4x^3 + 6x^2 - 16x + 14$.
Checking, $y = x^4 + 2x^3 - 8x^2 + 14x - 105$ so $y' = 4x^3 + 6x^2 - 16x + 14$.

66. $y = (x^4 + 3x^3 + 17x + 82)^3$, so $y' = 3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17)$.
You can expand y and check by differentiating directly yourself.

68. $g(x) = \begin{cases} -1-2x & \text{if } -\infty < x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } 1 < x < +\infty \end{cases}$

$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (-1-2x) = 1$,

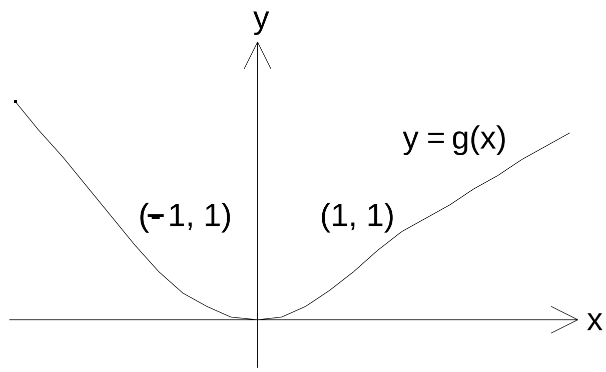
$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} x^2 = 1$, and $g(-1) = 1$,

so g is continuous at -1.

Also $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^2 = 1$,

$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} x = 1$, and $g(1) = 1$,

so g is continuous at 1.



For Exercise 68

Clearly g is continuous and differentiable everywhere in $(-\infty, -1)$ $(-1, 1)$ $(1, +\infty)$, since on each of these three intervals it coincides with a known differentiable function $-1-2x$, x^2, or x.
See graph of $y = g(x)$ above and to the right.

Now we must check differentiability at -1 and at 1, by evaluating left-hand and right-hand limits separately, using the definition of derivative.

$$\lim_{h \to 0^-} \frac{g(-1+h) - g(-1)}{h} = \lim_{h \to 0^-} \frac{[-1-2(-1+h)] - [-1]^2}{h} = \lim_{h \to 0^-} \frac{-2h}{h} = -2, \text{ and}$$

$$\lim_{h \to 0^+} \frac{g(-1+h) - g(-1)}{h} = \lim_{h \to 0^+} \frac{(-1+h)^2 - (-1)^2}{h} = \lim_{h \to 0^+} \frac{-2h+h^2}{h} = \lim_{h \to 0^+} (h-2) = -2 \text{ also,}$$

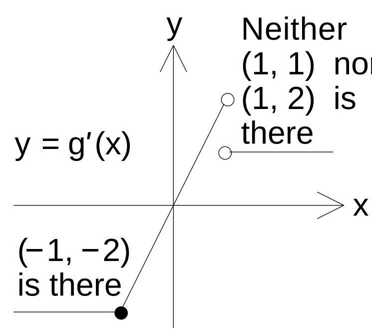
so $g'(-1)$ **does** exist, and $g'(-1) = -2$.

$$\lim_{h \to 0^-} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0^-} \frac{2h+h^2}{h} = \lim_{h \to 0^-} (2+h) = 2, \text{ while}$$

$$\lim_{h \to 0^+} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0^+} \frac{(1+h) - 1^2}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1,$$

so $g'(1)$ does **not** exist.

Thus $g'(x) = \begin{cases} -2 & \text{if } -\infty < x < -1 \\ -2 & \text{if } x = -1 \\ 2x & \text{if } -1 < x < 1 \\ 1 & \text{if } 1 < x < +\infty \end{cases}$



For Exercise 68

See graph of $y = g'(x)$ to the right.

The function $g'(x)$ has a serious discontinuity at $x = 1$; it is not a removable discontinuity since its left- and right-hand limits there disagree. This corresponds to the "corner" on the graph of $g(x)$ at $(1, 1)$; notice there is no corner at $(-1, 1)$.