MATHEMATICS 151

Assignment 6, due Monday 12 July 1999

Section 2.6 (pg. 162):

4. (a) If $\sqrt{x}+\sqrt{y}=4$, $\frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}=0$, and $y'=-\frac{\sqrt{y}}{\sqrt{x}}$.

(b) $y=\left(4-\sqrt{x}\right)^2=16-8\sqrt{x}+x$ so $y'=-\frac{4}{\sqrt{x}}+1$.

(c) $y'=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{4-\sqrt{x}}{\sqrt{x}}=-\frac{4}{\sqrt{x}}+1$ agreeing with (b).

10. If $\sqrt{x+y}+\sqrt{xy}=6$ then $\frac{1+y'}{2\sqrt{x+y}}+\frac{y+xy'}{2\sqrt{xy}}=0$, so $(1+y')\sqrt{xy}+(y+xy')\sqrt{x+y}=0$ and $y'=-\frac{\sqrt{xy}+y\sqrt{x+y}}{\sqrt{xy}+x\sqrt{x+y}}$.

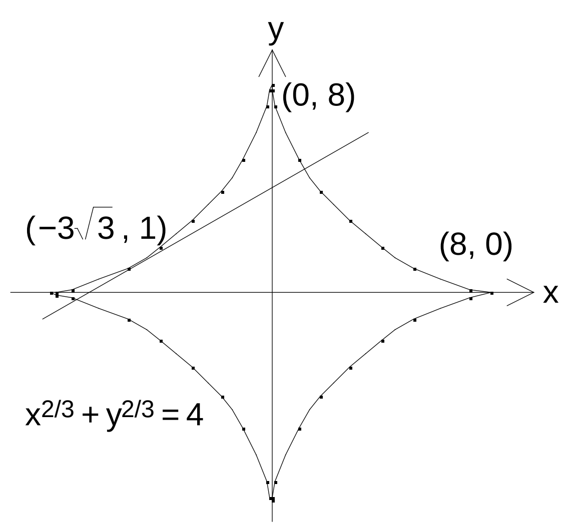
12. If $x\sqrt{1+y}+y\sqrt{1+2x}=2x$, then $\sqrt{1+y}+\frac{xy'}{2\sqrt{1+y}}+y'\sqrt{1+2x}+\frac{y}{\sqrt{1+2x}}=2$ and

$$y'=\frac{2-\sqrt{1+y}-\frac{y}{\sqrt{1+2x}}}{\frac{x}{2\sqrt{1+y}}+\sqrt{1+2x}}=\frac{4\sqrt{1+2x}\sqrt{1+y}-2(1+y)\sqrt{1+2x}-2y\sqrt{1+y}}{x\sqrt{1+2x}+2(1+2x)\sqrt{1+2y}}.$$

16. If $x\cdot(\cos y)+y\cdot(\cos x)=1$, $\cos y-xy'\cdot(\sin y)+y'\cdot(\cos x)-y\cdot(\sin x)=0$, and $y'=\frac{y\cdot(\sin x)-\cos y}{\cos x-x\cdot(\sin y)}$.

20. If $[g(x)]^2+12x=x^2g(x)$ and $g(4)=12$,
then $2g(x)g'(x)+12=2xg(x)+x^2g'(x)$,
so $2g(4)g'(4)+12=2\cdot4\cdot g(4)+4^2\cdot g'(4)$,
and $24g'(4)+12=96+16g'(4)$.
Thus $8g'(4)=84$, and $g'(4)=\frac{21}{2}$.

24. If $x^{2/3}+y^{2/3}=4$, then
$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$, and $y'=-\frac{y^{1/3}}{x^{1/3}}$.
At $(-3\sqrt{3},1)$, $y'=\frac{1}{\sqrt{3}}$. The tangent line at $(-3\sqrt{3},1)$ has equation
$y-1=\frac{1}{\sqrt{3}}\left(x+3\sqrt{3}\right)$, or $y=\frac{1}{\sqrt{3}}x+4$.
See graph above and to the right.



For Exercise 24

30. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$, and $y' = -\frac{xb^2}{ya^2}$. At (x_0, y_0), $y' = -\frac{x_0 b^2}{y_0 a^2}$.

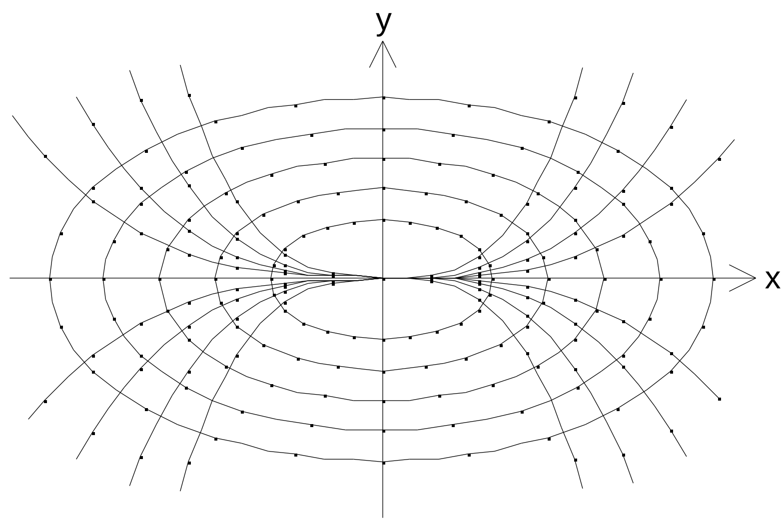
The tangent line at (x_0, y_0) has equation $y - y_0 = -\frac{x_0 b^2}{y_0 a^2}(x - x_0)$, or $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$.

But $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, so this tangent line has equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

40. If $y = ax^3$ then
$yx^{-3} = a$ so that
$y'x^{-3} - 3yx^{-4} = 0$.
So $y' = \frac{3y}{x}$ for these cubics.

If $x^2 + 3y^2 = b$ then
$2x + 6yy' = 0$, and thus
$y' = -\frac{x}{3y}$ for these ellipses.
Since the families' slopes $\frac{3y}{x}$ and $-\frac{x}{3y}$ are negative reciprocals, the families are orthogonal trajectories.

See graph to the right.



For Exercise 40

Section 2.7 (pg. 167):

2. The slope of the tangent line to the graph (c) is be zero at the origin. As we move to the right along the graph, this slope is maximized directly under the high point on the graph (b), but is everywhere positive, except at the beginning (the origin) and the end, where the slope is once more zero. The derivative of the function graphed in (c) is the function graphed in (b).

The slope of the tangent line to the graph (b) is positive at the origin, and increases for a while as we move to the right but then decreases. The transition point occurs directly under the high point on the graph (a). As we move to the right beyond the high point on the graph (b), the tangent line to the graph (b), which has previously had positive slope, now has negative slope and this is where the graph (a) drops below the t-axis. To the right of that point, the graph (b) has a tangent line with a slope becoming more and more negative as we move to the right, but then the slope begins to be less negative, although still negative. The transition occurs directly above the low point on the graph (a). At the right end, the slope of the tangent to the graph (b) has become zero, and the graph (a) meets the t-axis. The derivative of the function graphed in (b) is the function graphed in (a).

All this shows that (c) must graph the position function, (b) the velocity function, and (a) the acceleration function.

6. If $G(r) = \sqrt{r} + \sqrt[3]{r}$, $G'(r) = \frac{1}{2} r^{-1/2} + \frac{1}{3} r^{-2/3}$, and $G''(r) = -\frac{1}{4} r^{-3/2} - \frac{2}{9} r^{-5/3}$.

12. If $y = \frac{x^2}{x+1}$, $y' = \frac{(x+1)\cdot(2x) - x^2\cdot 1}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} = 1 - (x+1)^{-2}$ and

$y'' = \frac{(x+1)^2(2x+2) - (x^2+2x)\cdot 2(x+1)}{(x+1)^4} = \frac{2}{(x+1)^3} = 2(x+1)^{-3}$.

Alternatively, $y = x - 1 + \frac{1}{x+1} = x - 1 + (x-1)^{-1}$, so $y' = 1 - (x+1)^{-2}$, and $y'' = 2(x+1)^{-3}$.

18. If $y = \frac{1-x}{1+x}$ then $y' = \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = \frac{-2}{(1+x)^2} = -2(1+x)^{-2}$, $y'' = 4(1+x)^{-3}$, and $y''' = -12(1+x)^{-4}$.

Alternatively, $y = -1 + \frac{2}{1+x} = -1 + 2(1+x)^{-1}$, $y' = -2(1+x)^{-2}$, $y'' = 4(1+x)^{-3}$, and $y''' = -12(1+x)^{-4}$.

24. If $\sqrt{x} + \sqrt{y} = 1$, then $y^{1/2} = 1 - x^{1/2}$, $\frac{1}{2} x^{-1/2} + \frac{1}{2} y^{-1/2} y' = 0$, and $y' = -y^{1/2} x^{-1/2}$.

Consequently $y^{-1}(y')^2 = x^{-1}$. Differentiating $\frac{1}{2} x^{-1/2} + \frac{1}{2} y^{-1/2} y' = 0$, we see that $-\frac{1}{4} x^{-3/2} - \frac{1}{4} y^{-3/2}(y')^2 + \frac{1}{2} y^{-1/2} y'' = 0$. Replacing $y^{1/2}$ by $1 - x^{1/2}$ and $y^{-1}(y')^2$ by x^{-1},

$y'' = \frac{1}{2}\{x^{-3/2} + y^{-3/2}(y')^2\} y^{1/2} = \frac{1}{2}\{x^{-3/2} y^{1/2} + y^{-1}(y')^2\} = \frac{1}{2}\{x^{-3/2}(1 - x^{1/2}) + x^{-1}\} = \frac{1}{2} x^{-3/2}$.

30. $f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$, so $f'(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} = 2!(1-x)^{-3}$, $f''(x) = 2(-3)(1-x)^{-4}(-1) = 2\cdot 3(1-x)^{-4} = 3!(1-x)^{-4}$, $f'''(x) = 4!(1-x)^{-5}$, and in general $f^{(n)}(x) = (n+1)!(1-x)^{-(n+2)}$, $n \ge 0$.

34. $D(x\sin x) = x\cos x + \sin x$.
$D^2(x\sin x) = -x\sin x + 2\cos x$.
$D^3(x\sin x) = -x\cos x - 3\sin x$.
$D^4(x\sin x) = x\sin x - 4\cos x$.
$D^5(x\sin x) = x\cos x + 5\sin x$.

This pattern continues.

The first term of $D^n(x\sin x)$ is always $xD^n(\sin x)$, and the second is $nD^{(n-1)}(\sin x)$.
But $D^4(\sin x) = \sin x$, so $D^{34}(\sin x) = D^2(\sin x) = -\sin x$, $D^{35}(\sin x) = D^3(\sin x) = -\cos x$, and thus $D^{35}(x\sin x) = -x\cos x - 35\sin x$.

38. $s = 2t^3 - 7t^2 + 4t + 1$.

(a) $v = s' = 6t^2 - 14t + 4$ and $a = v' = s'' = 12t - 14$.

(b) $a(1) = -2$.

(c) When $v = 0$, $6t^2 - 14t + 4 = 0$ so $3t^2 - 7t + 2 = 0$, $(3t-1)(t-2) = 0$, and $t = 1/3$ or $t = 2$. Then $a(1/3) = -10$ and $a(2) = 10$.

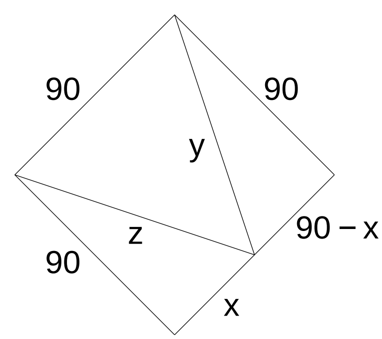
44. $Q(x) = ax^3 + bx^2 + cx + d$; $Q'(x) = 3ax^2 + 2bx + c$; $Q''(x) = 6ax + 2b$; $Q'''(x) = 6a$.
If $Q'''(1) = 12$, $6a = 12$ and $a = 2$. Then $Q''(x) = 12x + 2b$.
If $Q''(1) = 6$, $12 + 2b = 6$ and $b = -3$. Then $Q'(x) = 6x^2 - 6x + c$.
If $Q'(1) = 3$, $6 - 6 + c = 3$ and $c = 3$. Then $Q(x) = 2x^3 - 3x^2 + 3x + d$.
If $Q(1) = 1$, $2 - 3 + 3 + d = 1$, and $d = -1$. So $Q(x) = 2x^3 - 3x^2 + 3x - 1$.

Section 2.8 (pg. 172):

4. $x^2 + 3xy + y^2 = 1$, so $2x\frac{dx}{dt} + 3\frac{dx}{dt}y + 3x\frac{dy}{dt} + 2y\frac{dy}{dt} = 0$.
If $\frac{dy}{dt} = 2$, $(2x + 3y)\frac{dx}{dt} + 6x + 4y = 0$.
When $y = 1$, $x^2 + 3x = 0$ so $x = 0$ or $x = -3$.
Then $\frac{dx}{dt} = -\frac{6x+4y}{2x+3y} = -\frac{6x+4}{2x+3}$, so $\frac{dx}{dt} = -\frac{4}{3}$ or $\frac{dx}{dt} = -\frac{14}{3}$,
depending upon whether $x = 0$ or $x = -3$.

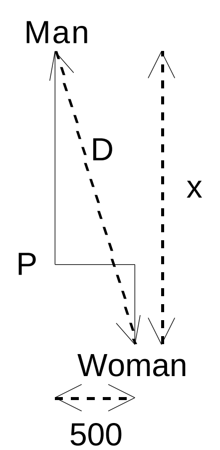
10. **(a)** If the batter has run x feet towards first base then his distance y from second base satisfies $y^2 = 90^2 + (90 - x)^2$, so $2y\frac{dy}{dt} = -2(90 - x)\frac{dx}{dt}$.
When he is halfway to first base, $x = 45$. But at that moment $y = 45\sqrt{5}$ and $\frac{dx}{dt} = 24$
so $\frac{dy}{dt} = -\frac{(90-x)}{y}\frac{dx}{dt} = -\frac{45}{45\sqrt{5}}\,24 = -\frac{24}{\sqrt{5}}$ and the distance to second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.73$ ft/s.

(b) At the same time if z is his distance from third base then $z^2 = x^2 + 90^2$ so $2z\frac{dz}{dt} = 2x\frac{dx}{dt}$.
When $x = 45$, $z = 45\sqrt{5}$ and $\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt} = \frac{45}{45\sqrt{5}}\,24 = \frac{24}{\sqrt{5}}$
so the distance to third base grows at $\frac{24}{\sqrt{5}} \approx 10.73$ ft/s.



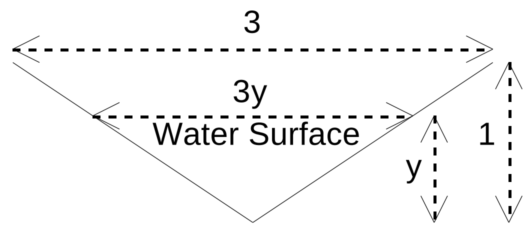
For Exercise 10

14. After the woman walks 15 min she is $5\cdot15\cdot60 = 4500$ ft south of her starting point and the man is $4\cdot(15 + 5)\cdot60 = 4800$ ft north of his starting point. If their north-south separation is x feet then the distance D between them satisfies $x^2 + 500^2 = D^2$, so
$2x\frac{dx}{dt} = 2D\frac{dD}{dt}$ and $\frac{dD}{dt} = \frac{x}{D}\frac{dx}{dt}$. But $\frac{dx}{dt} = 4 + 5 = 9$ and when
$x = 9300$ ft, $D = \sqrt{9300^2 + 500^2} = 100\sqrt{8674}$ ft. So at that time,
$\frac{dD}{dt} = \frac{9300}{100\sqrt{8674}}\,9 = \frac{837}{\sqrt{8674}} \approx 8.987$ ft/s. The man and woman are moving apart at a rate of $\frac{837}{\sqrt{8674}} \approx 8.987$ ft/s 15 minutes after the woman starts walking. This is only a little less than the sum of their individual speeds. This is reasonable, since by then she is 9300 ft south of him and only 500 ft east of him, so they are almost moving apart along a straight line.



For Exercise 14

18. Let the water level be y feet and let t be time, measured in minutes. When the water is y feet deep it will be $3y$ feet wide at its surface, the vertical cross-section area A of water will be $\frac{3}{2}y^2$ square feet, and the volume V of water in the trough will be $15y^2$ cubic feet.



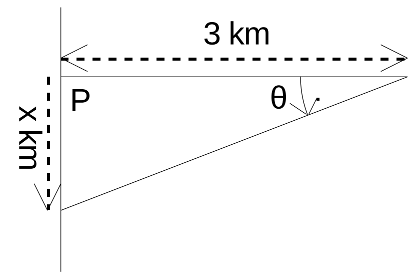
For Exercise 18

$\frac{dV}{dt} = 30y\frac{dy}{dt}$, so $30y\frac{dy}{dt} = 12$ and $\frac{dy}{dt} = \frac{2}{5y}$.

When $y = \frac{1}{2}$ ft, $\frac{dy}{dt} = \frac{4}{5}$ ft/min or 9.6 in/min.

The water level is rising at a rate of 9.6 in/min when the water depth is 6 in.

30. The light beam rotates at a rate of 8π radian/min. If θ is the angle between the light beam and the line segment from the lighthouse to the closest point P on the shore and if x is the distance in kilometres from P to the point where the beam meets the shore, then



For Exercise 30

$x = 3\tan\theta$ and $\frac{dx}{dt} = 3\sec^2\theta\frac{d\theta}{dt}$. If $x = 1$

then $\tan\theta = \frac{1}{3}$ so $\sec^2\theta = 1 + \tan^2\theta = \frac{10}{9}$ and

$\frac{dx}{dt} = 3\cdot\frac{10}{9}\cdot 8\pi = \frac{80}{3}\pi \approx 83.776$ km/min = 83776 m/min.

The beam of light is moving along the shoreline at a rate of $\frac{80}{3}\pi$ km/min or about 83776 m/min when it passes through a point 1 km from P.