MATHEMATICS 151

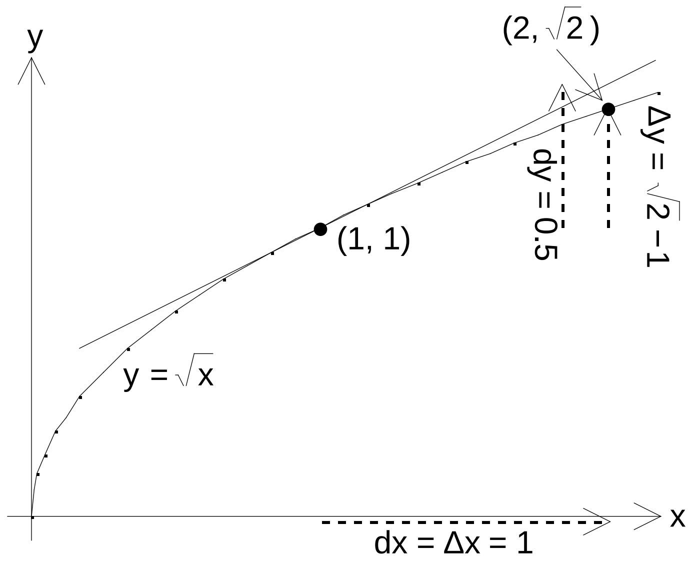
Assignment 7, due Friday 16 July 1999

Section 2.9 (pg. 180):

4. If $y = \frac{x-2}{2x+3}$ then
$dy = \frac{(2x+3)\cdot 1 - (x-2)\cdot 2}{(2x+3)^2}\,dx =$
$= \frac{7}{(2x+3)^2}\,dx.$

6. If $y = x(\tan x)$ then
$dy = [x(\sec^2 x) + \tan x]dx.$

10. If $y = \sqrt{1-x}$, then
$dy = -\frac{1}{2\sqrt{1-x}}\,dx$, so if $x = 0$
and $dx = 0.02$, $dy = -0.01$.



For Exercise 14

14. If $y = \sqrt{x}$, then $dy = \frac{1}{2\sqrt{x}}\,dx$.
If $x = 1$ and $dx = \Delta x = 1$, then $\Delta y = \sqrt{1+1} - \sqrt{1} = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \approx 0.414213562$
while $dy = \frac{1}{2}$. See graph above and to the right.

18. $y = x^4 + x^2 + 1$.
$dy = (4x^3 + 2x)\,dx =$
$= 6\,dx$ if $x = 1$.
See table to the right.

$dx = \Delta x$	Δy	dy	$\Delta y - dy$
1.0	18.0	6.0	12.0
0.5	5.3125	3.0	2.3125
0.1	0.6741	0.6	0.0741
0.01	0.06070401	0.06	0.00070401

22. Let $y = x^6$. Then $dy = 6x^5\,dx$. If $x = 2$ and $\Delta x = dx = -0.03$ then $dy = -5.76$ and $y + dy = 58.24$. In fact $1.97^6 = 58.451728309129$, exactly.

26. $A = \pi r^2$ so $dA = 2\pi r\,dr$.

(a) If $r = 24$ cm and $dr = 0.2$ cm then $dA = 9.6\pi \approx 30.16\text{ cm}^2$. (The maximum error can be a bit more than this; $\Delta A = \pi(24.2^2 - 24^2) = 9.64\pi \approx 30.28495318\text{ cm}^2$.)

(b) The relative error is $\frac{dA}{A} = \frac{9.6\pi}{576\pi} = \frac{1}{60} \approx 0.016666667$. Notice that the relative error in radius is $\frac{0.2}{24} = \frac{1}{120}$, so the relative error in area is exactly twice the relative error in radius. Can you explain why that **had** to be the case?

30. $V = \frac{2}{3}\pi r^3$; $r = 25$ m and $dr = 0.05\text{ cm} = 0.0005\text{m}$, so $dV = 2\pi r^2\,dr = \frac{5}{8}\pi$ or about 1.963 m^3.

Section 2.10 (pg. 185):

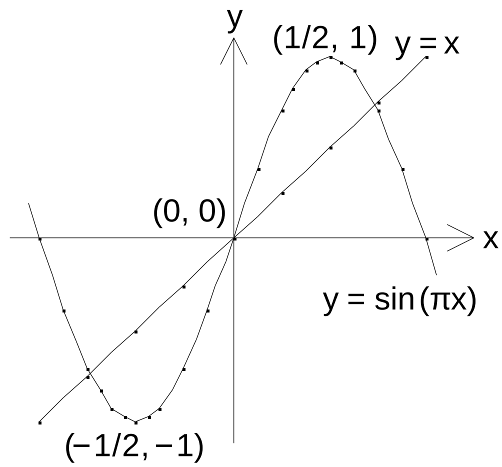
4. If $x^3 + x^2 + 2 = 0$, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2x_n^3 + x_n^2 - 2}{3x_n^2 + 2x_n}$.
If $x_1 = -2$, $x_2 = -1.75$, $x_3 \approx -1.6978021978$, $x_4 \approx -1.6956244765$, $x_5 \approx -1.6956207696$, and $x_6 \approx -1.6956207696$ also.
From here on, any changes have to be after the 10th decimal place.

6. If $x^7 - 100 = 0$, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{6x_n^7 + 100}{7x_n^6}$.
If $x_1 = 2$, $x_2 = 1.9375$, $x_3 \approx 1.9307689564$, $x_4 \approx 1.9306977368$, $x_5 \approx 1.9306977289$, and $x_6 \approx 1.9306977289$ also.
The nth root function on my Texas Instruments TI–36 calculator also gives $\sqrt[7]{100} \approx 1.9306977289$. The *Maple* program on my Macintosh computer gives $\sqrt[7]{100} \approx 1.9306977288832501670$, to 20 significant figures (more for the asking).

10. If $x^4 + x^3 - 22x^2 - 2x + 41 = 0$, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{3x^4 + 2x^3 - 22x^2 - 41}{4x^3 + 3x^2 - 44x - 2}$.
$f(1) = 19$ and $f(2) = -27$ so linear interpolation (pretending the graph is straight) leads to a first guess of about 1.4 for a root, since 27 is about $1.5 \cdot 19$. Of course you could try a different x_1; this time Newton's Method works for any x_1 in the interval $[1, 2]$.
$x_1 = 1.4$, $x_2 \approx 1.435632381$, $x_3 \approx 1.435476098$, $x_4 \approx 1.435476095$, $x_5 \approx 1.435476095$ gets the answer quickly to 10 decimal places. Starting with $x_1 = 1$ only takes one step longer. Starting with $x_1 = 2$ surprisingly is almost as fast as starting with $x_1 = 1.4$.
If you plot $f(x)$ (using *Maple*, for instance) over the interval $-1 \le x \le 3.5$ you will see why this works so well.

12. Let $f(x) = \tan x - x$. Note $f'(x) = \sec^2 x - 1 = \tan^2 x \ge 0$, so $f(x)$ is increasing in $(\pi/2, 3\pi/2)$. Since $f(4\pi/3) = \sqrt{3} - 4\pi/3 < 0$ while $\lim_{x \to 3\pi/2^-} f(x) = +\infty$, there is a root for the equation $f(x) = 0$ between $4\pi/3 \approx 4.188790205$ and $3\pi/2 \approx 4.712388980$.
Using $x_1 = 4.5$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan x_n - x_n}{\tan^2 x_n} = \frac{x_n - (\sin x_n)(\cos x_n)}{\sin^2 x_n}$,
$x_2 \approx 4.4936139028$, $x_3 \approx 4.4934096550$, $x_4 \approx 4.4934094579$, and $x_5 \approx 4.4934094579$ also.

18. Note $\frac{d}{dx}\sin(\pi x) = \pi\cos(\pi x)$, which is π when $x = 0$, so the graph of $y = \sin(\pi x)$ is steeper at the origin than that of $y = x$. After the peak at $(1/2, 1)$, $y = \sin(\pi x)$ drops while $y = x$ rises and the two graphs cross before $x = 1$ (since when $x = 1$, $\sin(\pi x) = 0$). After that, $y = x$ rises above $y = 1$, and $y = \sin(\pi x)$ can never get larger than 1. So there is a unique positive root for the equation $\sin(\pi x) = x$ between 1/2 and 1. (A good guess for it would be 0.75.) By symmetry there is a unique negative root between -1 and $-1/2$. And of course there is the root at $x = 0$.



If $f(x) = \sin(\pi x) - x$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin(\pi x_n) - x_n}{\pi\cos(\pi x_n) - 1} = \frac{\pi x_n \cos(\pi x_n) - \sin(\pi x_n)}{\pi\cos(\pi x_n) - 1}$, for $x_1 = 0.75$ we have $x_2 \approx 0.7366850852$, $x_3 \approx 0.7364844950$, $x_4 \approx 0.7364844482$, and $x_5 \approx 0.7364844482$. So the roots are $x = 0$ and $x \approx \pm 0.7364844482$.

See graph above and to the right.

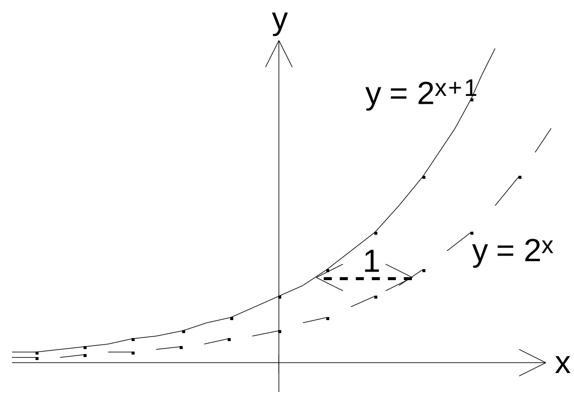
24. (a) Let $f(x) = \frac{1}{x} - a$. Then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = 2x_n - ax_n^2$.

(b) Since $\frac{1}{1.6984}$ is approximately 0.6, a good choice for x_1 is $x_1 = 0.6$. Then $x_2 \approx 0.5885760000$, $x_3 \approx 0.5887893715$, $x_4 \approx 0.5887894489$, and $x_5 \approx 0.5887894489$. So $\frac{1}{1.6984} \approx 0.5887894489$.

My TI–36 approximates $\frac{1}{1.6984}$ as 0.5887894489, as well.

Section 3.1 (pg. 200):

6. See graph to the right.



14. $\lim_{x \to -\infty} 1.1^x = 0$, since $1.1 > 1$.

18. $\lim_{x \to -\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \to -\infty} \frac{e^{6x} - 1}{e^{6x} + 1} = -1$.

26. On the next page are are tabulated values of $\frac{2.7^h - 1}{h}$ and $\frac{2.8^h - 1}{h}$ obtained by using a TI–36 calculator and rounding the results off to 5 decimal places.

h	1	0.1	0.01	0.001	0.0001	0.00001
$\frac{2.7^h - 1}{h}$	1.70000	1.04425	0.99820	0.99375	0.99330	0.99326
$\frac{2.8^h - 1}{h}$	1.80000	1.08449	1.03494	1.03015	1.02967	1.02962

h	−1	−0.1	−0.01	−0.001	−0.0001	−0.00001
$\frac{2.7^h - 1}{h}$	0.62963	0.94552	0.98834	0.99276	0.99320	0.99325
$\frac{2.8^h - 1}{h}$	0.64286	0.97839	1.02437	1.02909	1.02957	1.02961

Apparently, to two decimal places, $\lim_{h \to 0} \frac{2.7^h - 1}{h} \approx 0.99$ and $\lim_{h \to 0} \frac{2.8^h - 1}{h} \approx 1.03$.
This tells us that e lies between 2.7 and 2.8, probably closer to 2.7.

28. If $f(x) = xe^{-x^2}$, then $f'(x) = 1 \cdot e^{-x^2} + x \cdot \left[e^{-x^2} \cdot (-2x)\right] = (1 - 2x^2)e^{-x^2}$.

32. If $h(\theta) = e^{\sin(5\theta)}$, $h'(\theta) = 5\cos(5\theta)e^{\sin(5\theta)}$.

38. If $y = \sqrt[3]{2x + e^{3x}} = (2x + e^{3x})^{1/3}$, $y' = \frac{1}{3}(2x + e^{3x})^{-2/3}(2 + 3e^{3x}) = \frac{2 + 3e^{3x}}{3\sqrt[3]{(2x + 3^{3x})^2}}$.

42. If $y = \sec\left(e^{\tan(x^2)}\right)$, $y' = 2x\sec^2(x^2)e^{\tan(x^2)} \sec\left(e^{\tan(x^2)}\right) \tan\left(e^{\tan(x^2)}\right)$.

48. If $y = Ae^{-x} + Bxe^{-x} = (A + Bx)e^{-x}$, then using the product and chain rules,
$y' = [B + (A + Bx)(-1)]e^{-x} = [(B - A) - Bx]e^{-x}$.
Likewise $y'' = \{-B + [(B - A) - Bx](-1)\}e^{-x} = [(A - 2B) + Bx]e^{-x}$.
So $y'' + 2y' + y = [(A - 2B) + Bx]e^{-x} + 2[(B - A) - Bx]e^{-x} + (A + Bx)e^{-x} =$
$= \{[(A - 2B) + 2(B - A) + A] + [Bx - 2Bx + Bx]\}e^{-x} = 0$.

52. If $f(x) = xe^{-x}$ then $f'(x) = [1 + x \cdot (-1)]e^{-x} = [1 - x]e^{-x}$ (product and chain rules).
Then $f''(x) = [(-1) + (1 - x) \cdot (-1)]e^{-x} = [-2 + x]e^{-x}$.
Likewise $f'''(x) = [1 + (-2 + x) \cdot (-1)]e^{-x} = [3 - x]e^{-x}$.
So it appears that $f^{(n)}(x) = [(-1)^{n+1}n + (-1)^n x]e^{-x} = (-1)^n(x - n)e^{-x}$.
Students familiar with Mathematical Induction (Appendix E in the text) can verify this.
The formula certainly is true when $n = 0$ and when $n = 1$.
If it is true when $n = k$, so that $f^{(k)}(x) = (-1)^k(x - n)e^{-x}$, by the product and chain rules that will make $f^{(k+1)}(x) = (-1)^k[1 + (x - n) \cdot (-1)]e^{-x} = (-1)^{k+1}[x - (n + 1)]e^{-x}$.
In other words, if the formula is ever true, it will be true the next time too.
Since it is true at the beginning ($n = 0$), it stays true forever.
That makes $f^{(1000)}(x) = (x - 1000)e^{-x}$.