MATHEMATICS 151

Assignment 8, due Monday 19 July 1999

Section 3.2 (pg. 208):

2. f is one-to-one since horizontal lines never meet the graph more than once.

4. f is not one-to-one since there is a horizontal line which meets the graph more than once—in fact, in infinitely many places along a line segment.

6. f is one-to-one since horizontal lines never meet the graph more than once.

8. $f(x) = x^2 - 2x + 5 = (x-1)^2 + 4$ is not one-to-one, since $f(1+h) = f(1-h) = h^2 + 4$.

10. $g(x) = |x|$ is not one-to-one since $g(-x) = g(x)$.

16. If $f(x) = 5 - 4x^3$ then f is decreasing and can never take the same value twice. Write $y = 5 - 4x^3$ and solve for $x = \sqrt[3]{\frac{5-y}{4}}$.

Interchanging x and y, the inverse function is $y = f^{-1}(x) = \sqrt[3]{\frac{5-x}{4}}$.

18. If $f(x) = x^2 + x = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$, $x \ge -\frac{1}{2}$, f increases and never takes the same value twice. Write $y = x^2 + x$ and solve for $x = \frac{-1 \pm \sqrt{1^2 + 4y}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4y}$.

But we must have $x \ge -\frac{1}{2}$, so $x = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4y}$.

Interchanging x and y, the inverse function is $y = f^{-1}(x) = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4x}$.

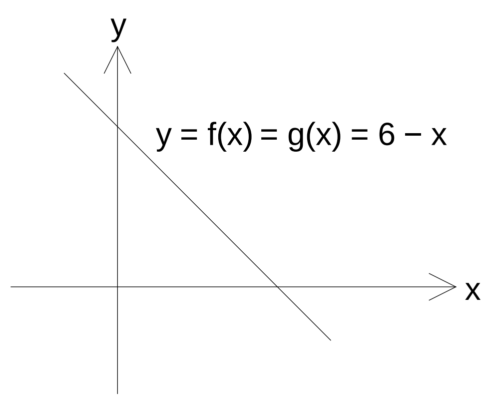
20. (a) $f(x) = 6 - x$ is decreasing hence one-to-one. The domain and range are $(-\infty, +\infty)$.

(b) $f'(x) = -1$ hence $g'(a) = \frac{1}{f'(g(a))} = -1$. If $a = 2$, $g'(2) = -1$. Compare with (d).

(c) If $y = 6 - x$, $x = 6 - y$. Interchanging x and y, $f^{-1}(x) = g(x) = 6 - x$. The domain and range of g are both $(-\infty, +\infty)$, the same as the range and domain of f.

(d) Since $g(x) = 6 - x$, $g'(x) = -1$ and $g'(2) = -1$. Compare with (b).

(e) See graphs of f and g to the right.



For Exercise 20

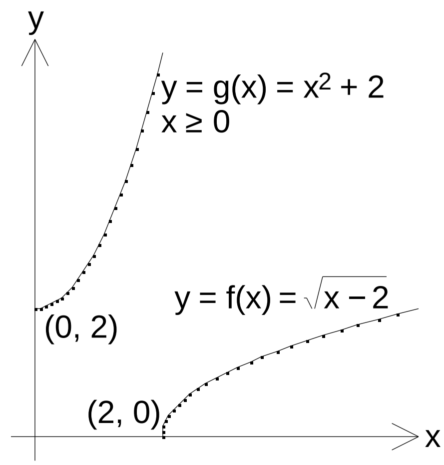
22. (a) $f(x) = \sqrt{x-2}$ is increasing.
Hence it is one-to-one.
Its domain is $[2, +\infty)$; its range is $[0, +\infty)$.

(b) $f'(x) = \frac{1}{2\sqrt{x-2}}$, so $g'(a) = 2\sqrt{g(a)-2}$.
Since $f(6) = 2$, $g(2) = 6$.
So $g'(2) = 2\sqrt{6-2} = 4$. Compare with (d).

(c) If $y = \sqrt{x-2}$, $x = y^2 + 2$.
Interchanging x and y, $f^{-1}(x) = g(x) = x^2 + 2$.
The domain of g is $[0, +\infty)$, the range of f;
the range of g is $[2, +\infty)$, the domain of f.

(d) $g'(x) = 2x$ and $g'(2) = 4$.
Compare with (b).

(e) See graphs of f and g to the right.



For Exercise 22

40. $\sin(x + 2\pi) = \sin x$, hence $h(x) = \sin x$ is not one-to-one on $(-\infty, +\infty)$.
$h(x) = \sin x$ is increasing on $[-\pi/2, \pi/2]$, hence it is one-to-one there.
Let $y = \sin^{-1} x$, so $x = \sin y$. Differentiating with respect to x, $1 = (\cos y)\cdot\frac{dy}{dx}$, so
$\frac{d}{dx}\sin^{-1}x = \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$ provided $\cos(\sin^{-1}x) \neq 0$, i.e. provided $-1 < x < 1$. (Note that $1 = \cos^2(\sin^{-1}x) + \sin^2(\sin^{-1}x) = \cos^2(\sin^{-1}x) + x^2$, so $\cos^2(\sin^{-1}x) = 1 - x^2$ and $\cos(\sin^{-1}x) = \pm\sqrt{1-x^2}$. But since $-\pi/2 \le \sin^{-1}x \le \pi/2$, $\cos(\sin^{-1}x) \ge 0$ and the $+$ sign must be correct.)

Section 3.3 (pg. 214):

4. $\log_8 4 = \frac{2}{3}$ since $8^{2/3} = 4$.

10. $\log_3 108 - \log_3 4 = \log_3 \frac{108}{4} = \log_3 27 = 3$ since $3^3 = 27$.

14. $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$.

36. $2^{x-5} = 3$ is equivalent to $x - 5 = \log_2 3 = \frac{\ln 3}{\ln 2}$, or to $x = 5 + \log_2 3 = 5 + \frac{\ln 3}{\ln 2}$.

46. $\ln x + \ln(x-1) = 1$ is equivalent to $\ln[x(x-1)] = 1$, $x > 0$, $x - 1 > 0$. (The inequalities must be included, since it is possible for to have $x < 0$, $x - 1 < 0$, but $x(x-1) > 0$, and then x and $x - 1$ do not have logarithms, while $x(x-1)$ does.)
Thus we solve $x(x-1) = e$, obtaining $x = \frac{1 \pm \sqrt{1+4e}}{2} = \frac{1}{2} \pm \frac{\sqrt{1+4e}}{2}$. However we must

reject $\frac{1}{2}-\frac{\sqrt{1+4e}}{2}$ since it is negative and has no logarithm. The only solution is
$x=\frac{1+\sqrt{1+4e}}{2}=\frac{1}{2}+\frac{\sqrt{1+4e}}{2}$.

66. $\lim_{x \to 0^+} \ln(\sin x) = -\infty$, since $\lim_{x \to 0^+} \sin x = 0$, the limiting value of $\sin x$ being approached from the right so that $\ln(\sin x)$ is meaningful when x is close enough to 0 but on the right.

68. $\lim_{x \to +\infty} \frac{\ln x}{1 + \ln x} = \lim_{x \to +\infty} \frac{1}{\frac{1}{\ln x} + 1} = 1$.

74. $t^3 - t = t(t+1)(t-1) > 0$ on $(-1, 0) \quad (1, +\infty)$ so the domain of $G(t) = \ln(t^3 - t)$ is $(-1, 0) \quad (1, +\infty)$. But $t^3 - t = t(t^2 - 1)$ takes on all values in $(0, 2\sqrt{3}/9]$ for $-1 < t < 0$ and all values in $(0, +\infty)$ for $0 < t < +\infty$, so the range of $G(t) = \ln(t^3 - t)$ is $(-\infty, +\infty)$.

80. $y = \frac{1 + e^x}{1 - e^x}$ with domain $(-\infty, 0) \quad (0, +\infty)$.
$y' = \frac{(1 - e^x)\cdot e^x - (1 + e^x)\cdot(-e^x)}{(1 - e^x)^2} = \frac{2e^x}{(1 - e^x)^2} > 0$ so the function is increasing on $(-\infty, 0)$ and on $(0, +\infty)$. Since $\lim_{x \to -\infty} \frac{1 + e^x}{1 - e^x} = 1$, $\lim_{x \to 0^-} \frac{1 + e^x}{1 - e^x} = +\infty$, $\lim_{x \to 0^+} \frac{1 + e^x}{1 - e^x} = -\infty$,
and $\lim_{x \to +\infty} \frac{1 + e^x}{1 - e^x} = \lim_{x \to +\infty} \frac{e^{-x} + 1}{e^{-x} - 1} = -1$, the range of our function is $(-\infty, -1) \quad (1, +\infty)$, the values in $(-\infty, -1)$ occurring when $0 < x < +\infty$ and the values in $(1, +\infty)$ occurring when $-\infty < x < 0$.
$y = \frac{1 + e^x}{1 - e^x} \Leftrightarrow y - ye^x = 1 + e^x \Leftrightarrow e^x(y + 1) = y - 1 \Leftrightarrow e^x = \frac{y - 1}{y + 1} \Leftrightarrow x = \ln \frac{y - 1}{y + 1}$.
So the inverse function has $y = \ln \frac{x - 1}{x + 1}$, with domain $(-\infty, -1) \quad (1, +\infty)$ and range $(-\infty, 0) \quad (0, +\infty)$.

Section 3.4 (pg. 221):

2. $f(x) = \cos(\ln x)$ with domain $(0, +\infty)$, so $f'(x) = -\frac{\sin(\ln x)}{x}$ with domain $(0, +\infty)$.

6. $f(x) = \sqrt{3 - 2^x}$ has domain $(-\infty, \log_2 3] = (-\infty, (\ln 3)/(\ln 2)]$.
$f'(x) = \frac{1}{2\sqrt{3 - 2^x}}(-2^x \ln 2) = -\frac{2^{x-1}\ln 2}{\sqrt{3 - 2^x}}$ has domain $(-\infty, \log_2 3) = (-\infty, (\ln 3)/(\ln 2))$.

8. If $y = \ln(ax)$, $y' = \frac{1}{ax}\cdot a = \frac{1}{x}$ and $y'' = -\frac{1}{x^2}$, assuming $a \neq 0$.

Alternatively, $y = \ln a + \ln x$ so $y' = 0 + \frac{1}{x} = \frac{1}{x}$. However this argument only holds for $a > 0$ and $x > 0$. If $a < 0$ and $x < 0$ then $y = \ln(ax) = \ln[(-a)(-x)] = \ln(-a) + \ln(-x)$ and then $y' = 0 + \frac{1}{-x}\cdot(-1) = \frac{1}{x}$.

10. If $y = \ln(\sec x + \tan x)$, $y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$ and $y'' = \sec x \tan x$.

20. $G(u) = \ln\sqrt{\frac{3u+2}{3u-2}} = \ln\left(\frac{3u+2}{3u-2}\right)^{1/2}$.

$$G'(u) = \frac{1}{\left(\frac{3u+2}{3u-2}\right)^{1/2}}\ \frac{1}{2}\left(\frac{3u+2}{3u-2}\right)^{-1/2}\ \frac{(3u-2)\,3-(3u+2)\,3}{(3u-2)^2} = -\frac{6}{(3u+2)(3u-2)} = -\frac{6}{9u^2-4}.$$

28. $G(x) = 5^{\tan x}$ so $G'(x) = 5^{\tan x}\cdot\ln 5\cdot\sec^2 x$.

42. $y = (\sin x)^{\cos x} = e^{\cos x\cdot\ln(\sin x)}$.

$y' = e^{\cos x\cdot\ln(\sin x)}\cdot\left[-(\sin x)\cdot\ln(\sin x) + \cos x\cdot\frac{1}{\sin x}\cdot\cos x\right] =$

$= (\sin x)^{\cos x}\left[-(\sin x)\cdot\ln(\sin x) + \frac{\cos^2 x}{\sin x}\right]$.

Alternatively, $\ln y = (\cos x)\ln(\sin x)$.

$\frac{y'}{y} = -(\sin x)\cdot\ln(\sin x) + \cos x\,\frac{\cos x}{\sin x} = -(\sin x)\cdot\ln(\sin x) + \frac{\cos^2 x}{\sin x}$.

$y' = y\left[-(\sin x)\cdot\ln(\sin x) + \frac{\cos^2 x}{\sin x}\right] = (\sin x)^{\cos x}\left[-(\sin x)\cdot\ln(\sin x) + \frac{\cos^2 x}{\sin x}\right]$.

60. $y = \sqrt{\frac{x^2+1}{x+1}}$, $x > -1$, so $\ln y = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x+1)$.

$\frac{y'}{y} = \frac{x}{x^2+1} - \frac{1}{2(x+1)}$.

$$y' = y\left[\frac{x}{x^2+1} - \frac{1}{2(x+1)}\right] = \frac{(x^2+1)^{1/2}}{(x+1)^{1/2}}\left[\frac{x}{x^2+1} - \frac{1}{2(x+1)}\right] = \frac{x}{(x+1)^{1/2}(x^2+1)^{1/2}} - \frac{(x^2+1)^{1/2}}{2(x+1)^{3/2}}.$$

62. $y = \frac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}$, so $\ln y = 4\ln(x^3+1) + 2\ln(\sin x) - \frac{1}{3}\ln x$, $x > 0$.

$\frac{y'}{y} = \frac{12x^2}{x^3+1} + 2\,\frac{\cos x}{\sin x} - \frac{1}{3x}$.

$$y' = y\left[\frac{12x^2}{x^3+1} + 2\frac{\cos x}{\sin x} - \frac{1}{3x}\right] = \frac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}\left[\frac{12x^2}{x^3+1} + 2\frac{\cos x}{\sin x} - \frac{1}{3x}\right] =$$

$= 12(x^3+1)^3x^{5/3}\sin^2 x + 2(x^3+1)^4x^{-1/3}\cos x\sin x - \frac{1}{3}x^{-4/3}(x^3+1)^4\sin^2 x$.