MATHEMATICS 151

Assignment 9, due Wednesday 21 July 1999

Section 3.5 (pg. 226):

2. (a) In t hours there will be $2t$ half-hours.
Thus the number of bacteria after t hours will be $N(t) = 4000 \cdot 3^{2t} = 4000 \cdot 9^{t}$.
Alternatively $N(t) = 4000 \cdot e^{(\ln 3 \cdot 2)t} = 4000 \cdot e^{(\ln 9)t}$.

(b) In 20 minutes there will be $N(1/3) = 4000 \cdot 3^{2/3}$ or $4000 \cdot e^{(\ln 9)/3}$ bacteria.
But $4000 \cdot 3^{2/3} = 4000 \cdot e^{(\ln 9/3)} \approx 8320.335292$. So there will be 8320 bacteria.

(c) If $N(t) = 20000$ then $3^{2t} = 5$ and $2t = \frac{\ln 5}{\ln 3}$ so $t = \frac{1}{2}\frac{\ln 5}{\ln 3} \approx 0.73248676$.
This will take about 0.73 hours, or about 44 minutes.

4. $N(t) = A \cdot 2^{kt}$, with $400 = A \cdot 2^{2k}$ and $25600 = A \cdot 2^{6k}$.
Cubing $400 = A \cdot 2^{2k}$, $64000000 = A^3 \cdot 2^{6k}$, so $2500 = A^2$, $A = 50$, and $512 = 2^{6k}$.
But $512 = 2^9$, so $6k = 9$ and $k = 1.5$.

(a) The initial size of the culture was 50.

(b) $N(t) = 50 \cdot 2^{1.5t}$.

(c) $1.5t = 1$ when $t = 2/3$. The population doubles in 2/3 hour, or 40 minutes.

(d) When $N(t) = 100000$, $50 \cdot 2^{1.5t} = 100000$, $2^{1.5t} = 2000$, $1.5t = \frac{\ln 2000}{\ln 2}$,
and $t = \frac{2}{3}\frac{\ln 2000}{\ln 2} \approx 7.310522856$.
So we must wait about 7.3105 hours, or about 7 hours 18 minutes 38 seconds.

Alternatively, write $N(t) = A \cdot e^{ct}$, with $400 = A \cdot e^{2c}$ and $25600 = A \cdot e^{6c}$.
Then $64000000 = A^3 e^{6c}$, so $A^2 = 2500$ and $A = 50$. Hence $e^{2c} = 8$, and $c = \frac{\ln 8}{2}$.

(a) The initial size of the culture was 50.

(b) $N(t) = 50 \cdot e^{((t\ln 8)/2)}$.

(c) If $\frac{t \ln 8}{2} = \ln 2$, then $t = \frac{2 \ln 2}{\ln 8} = \frac{2 \ln 2}{3 \ln 2} = \frac{2}{3}$.
So the population doubles in 2/3 hour, or 40 minutes.

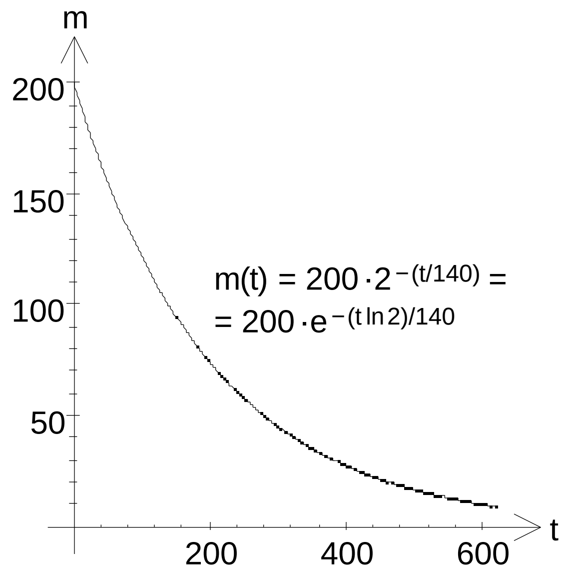
(d) If $N(T) = 100000$, $e^{((t\ln 8)/2)} = 2000$; $t = \frac{2 \ln 2000}{\ln 8} \approx 7.310522856$, as before.

8. (a) If the original amount is 200 mg and the half-life is 140 days, then the mass at time t is $m(t) = 200 \cdot 2^{-(t/140)}$.
Alternatively $m(t) = 200 \cdot e^{-ct}$, where $100 = 200 \cdot e^{-(140c)}$, so $2 = e^{140c}$ and $c = \frac{\ln 2}{140}$.
Thus $m(t) = 200 \cdot e^{-(t\ln 2)/140}$.

(b) After 100 days, $m(100) = 200 \cdot 2^{-100/140} = 200 \cdot 2^{-5/7} \approx 121.9013654$.
Alternatively $m(100) = 200 \cdot e^{-(100\ln 2)/140} \approx 121.9013654$. About 121.9 mg will be left.

(c) When $m(t) = 10$,
$10 = 200\cdot 2^{-t/140}$ so $2^{t/140} = 20$
and $t = 140\frac{\ln 20}{\ln 2} \approx 605.0699333$.
Alternatively $10 = 200\cdot e^{-(t\ln 2)/140}$
so $\frac{t\ln 2}{140} = \ln 20$ and $t = 140\frac{\ln 20}{\ln 2}$.
We will have to wait about 605 days.

(d) See graph to the right.



For Exercise 8

14. If T is the temperature indicated on the thermometer, measured in degrees Celsius, and t is the time in minutes after the thermometer is taken outdoors, then
$T - 5 = 15\cdot e^{-ct}$, where $12 - 5 = 15\cdot e^{-c\cdot 1}$.
Then $e^c = \frac{15}{7}$, and $c = \ln\frac{15}{7}$, so that
$T = 5 + 15\cdot e^{-(t\ln(15/7))} = 5 + 15\cdot\left(\frac{7}{15}\right)^t$.

(a) After 1 more minute, when $t = 2$,
$T = 5 + 15\cdot e^{-(2\ln(15/7))} = 5 + 15\cdot\left(\frac{7}{15}\right)^2 = 5 + \frac{49}{15} = \frac{124}{15} \approx 8.266666667°$ C.

(b) When $T = 6°$ C, $6 = 5 + 15\left(\frac{7}{15}\right)^t$ and $t\ln\frac{7}{15} = -\ln 15$, so
$t = \frac{\ln 15}{\ln 15 - \ln 7} \approx 3.55321859$ minutes, or about 3'33".

18. (a) With annual compounding, after two years the amount due will be
$500\cdot 1.14^2 = 649.8$, or \$649.80.

(b) With quarterly compounding, after two years the amount due will be
$500\cdot\left(1 + \frac{0.14}{4}\right)^8 = 500\cdot 1.035^8 \approx 658.4045185$, or \$658.40.

(c) With monthly compounding, after two years the amount due will be
$500\cdot\left(1 + \frac{0.14}{12}\right)^{24} \approx 660.4935501$, or \$660.49.

(d) With daily compounding (barring leap years), after two years the amount due will be $500\cdot\left(1 + \frac{0.14}{365}\right)^{730} \approx 661.5293892$, or \$661.53.

(e) With continuous compounding, after two years the amount due will be
$500\cdot e^{2\cdot 0.14} \approx 661.5649062$, or \$661.56.

Section 3.6 (pg. 233):

2. $\sin^{-1}(0.5) = \pi/6$ since $\sin(\pi/6) = 0.5$ and $-\pi/2 \le \pi/6 \le \pi/2$.

4. $\arctan(-1) = -\pi/4$ since $\tan(-\pi/4) = -1$ and $-\pi/2 < -\pi/4 < \pi/2$.

14. $\tan(\arctan 2) = 2$, so $\sec^2(\arctan 2) = 1 + \tan^2(\arctan 2) = 1 + 2^2 = 5$.
But $-\pi/2 < \arctan 2 < \pi/2$, so $\sec(\arctan 2) > 0$, and $\sec(\arctan 2) = \sqrt{5}$.

18. $\sin(\sin^{-1}(1/3)) = 1/3$, so $\cos(\sin^{-1}(1/3)) = \sqrt{1-(1/3)^2} = 2\sqrt{2}/3$.
Likewise $\sin(\sin^{-1}(2/3)) = 2/3$, so $\cos(\sin^{-1}(2/3)) = \sqrt{1-(2/3)^2} = \sqrt{5}/3$. Thus
$\sin\left[\sin^{-1}(1/3) + \sin^{-1}(2/3)\right] = \sin(\sin^{-1}(1/3))\cos(\sin^{-1}(2/3)) + \cos(\sin^{-1}(1/3))\sin(\sin^{-1}(2/3)) =$
$= \frac{1}{3}\,\frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}\,\frac{2}{3} = \frac{\sqrt{5}+4\sqrt{2}}{9}$.

22. $\sin^2(\cos^{-1}x) = 1 - \cos^2(\cos^{-1}x) = 1 - x^2$. But $0 \le \cos^{-1}x \le \pi$, so $\sin(\cos^{-1}x) \ge 0$,
$\sin(\cos^{-1}x) = \sqrt{1-x^2}$, and $\sin(2\cos^{-1}x) = 2\cos(\cos^{-1}x)\sin(\cos^{-1}x) = 2x\sqrt{1-x^2}$.

30. If $f(x) = \sin^{-1}(2x-1)$, then $f'(x) = \frac{1}{\sqrt{1-(2x-1)^2}}\,2 = \frac{2}{\sqrt{4x-4x^2}} = \frac{1}{\sqrt{x-x^2}}$.

38. If $F(t) = \sqrt{1-t^2} + \sin^{-1}t$, then $F'(t) = \frac{1}{2}\,\frac{(-2t)}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} = \frac{1-t}{\sqrt{1-t^2}} = \sqrt{\frac{1-t}{1+t}}$.

48. If $y = x\sin x\csc^{-1}x$, then $y' = \sin x\csc^{-1}x + x\cos x\csc^{-1}x + x\sin x\left(-\frac{1}{x\sqrt{x^2-1}}\right) =$
$= \sin x\csc^{-1}x + x\cos x\csc^{-1}x - \frac{\sin x}{\sqrt{x^2-1}}$.

64. $\lim_{x\to+\infty}(x - x^2) = \lim_{x\to+\infty} x(1-x) = -\infty$, so $\lim_{x\to+\infty}\tan^{-1}(x-x^2) = -\pi/2$.