MATHEMATICS 151

Assignment 10, due Friday 23 July 1999

Section 3.8 (pg. 247):

2. $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1} = \lim\limits_{x \to 1} \dfrac{2x + 3}{1} = 5.$

Or avoid L'Hospital's Rule with $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x-1)(x+4)}{x - 1} = \lim\limits_{x \to 1} (x + 4) = 5.$

10. $\lim\limits_{x \to 3\pi/2} \dfrac{\cos x}{x - \frac{3\pi}{2}} = \lim\limits_{x \to 3\pi/2} \dfrac{-\sin x}{1} = 1.$

Or avoid L'Hospital's Rule with $\lim\limits_{x \to 3\pi/2} \dfrac{\cos x}{x - \frac{3\pi}{2}} = \lim\limits_{x \to 3\pi/2} \dfrac{\sin\left(x - \frac{3\pi}{2}\right)}{x - \frac{3\pi}{2}} = 1.$

16. $\lim\limits_{x \to 0} \dfrac{6^x - 2^x}{x} = \lim\limits_{x \to 0} \dfrac{6^x \ln 6 - 2^x \ln 2}{1} = \ln 6 - \ln 2 = \ln \dfrac{6}{2} = \ln 3.$

Or avoid L'Hospital's Rule in the following way.

Let $f(x) = 6^x - 2^x$. Then $f'(x) = 6^x \ln 6 - 2^x \ln 2$, so $f'(0) = \ln 6 - \ln 2 = \ln \dfrac{6}{2} = \ln 3.$

But $\lim\limits_{x \to 0} \dfrac{6^x - 2^x}{x} = \lim\limits_{x \to 0} \dfrac{(6^x - 2^x) - (6^0 - 2^0)}{x - 0} = f'(0)$, from the definition of the derivative.

22. $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^3} = \lim\limits_{x \to 0} \dfrac{\cos x - 1}{3x^2} = \lim\limits_{x \to 0} \dfrac{-\sin x}{6x} = \lim\limits_{x \to 0} \dfrac{-\cos x}{6} = -\dfrac{1}{6}.$

30. $\lim\limits_{x \to 0} \dfrac{\sin(mx)}{\sin(nx)} = \lim\limits_{x \to 0} \dfrac{m\cos(mx)}{n\cos(nx)} = \dfrac{m}{n}.$

Or avoid L'Hospital's Rule by writing $\lim\limits_{x \to 0} \dfrac{\sin(mx)}{\sin(nx)} = \lim\limits_{x \to 0} \dfrac{\frac{\sin(mx)}{mx}}{\frac{\sin(nx)}{nx}} \cdot \dfrac{m}{n} = \dfrac{1}{1} \cdot \dfrac{m}{n} = \dfrac{m}{n}.$

44. $\lim\limits_{x \to 0^+} \sqrt{x} \sec x = 0\ 1 = 0.$ L'Hospital's Rule is inapplicable.

48. $\lim\limits_{x \to 0} (\csc x - \cot x) = \lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x} = \lim\limits_{x \to 0} \dfrac{\sin x}{\cos x} = 0.$

56. Let $y = (\sin x)^{(\tan x)}$, $0 < x < \pi/2$. Then $\ln y = (\tan x)(\ln(\sin x))$.

So $\lim\limits_{x \to 0^+} \ln y = \lim\limits_{x \to 0^+} \dfrac{\ln(\sin x)}{\cot x} = \lim\limits_{x \to 0^+} \dfrac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim\limits_{x \to 0^+} (-\sin x \cos x) = 0.$

Hence $\lim\limits_{x \to 0^+} y = e^0 = 1.$

78. $\lim\limits_{x \to +\infty} \dfrac{\ln x}{x^p} = \lim\limits_{x \to +\infty} \dfrac{x^{-1}}{p x^{p-1}} = \lim\limits_{x \to +\infty} \dfrac{1}{p x^p} = 0$, if $p > 0$.

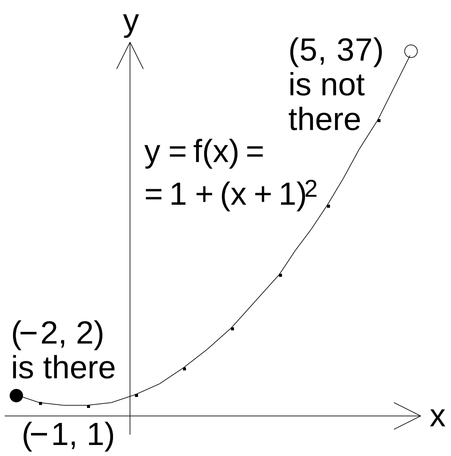
Section 4.1 (pg. 260):

12. $f(x) = 1 + (x + 1)^2$, $-2 \le x < 5$. $f'(x) = 2(x + 1)$, $-2 < x < 5$ so $f'(x) = 0$ at $x = -1$. $f(-2) = 2$, $f(-1) = 1$, and since $(x + 1)^2 \ge 0$, $f(x) = 1 + (x + 1)^2 \ge 1$. The absolute minimum is $f(-1) = 1$. Since $\lim_{x \to 5^-} f(x) = 37 > 2 = f(-2)$, there is no absolute maximum.

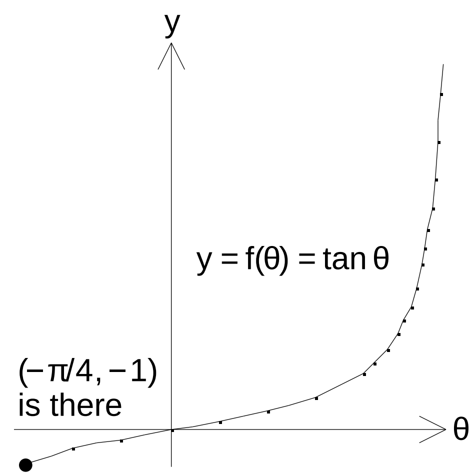
See graph below.

16. $f(\theta) = \tan\theta$, $-\pi/4 \le \theta < \pi/2$. $f'(\theta) = \sec^2\theta$, $-\pi/4 < \theta < \pi/2$. $f'(\theta) = 0$ **never**. Since $f(-\pi/4) = -1 \le f(\theta)$ for all $\theta \quad [-\pi/4, \pi/2)$ and $\lim_{\theta \to \pi/2^-} f(\theta) = = +\infty$, the absolute minimum is $f(-\pi/4) = -1$ and there is no absolute maximum.

See graph below.



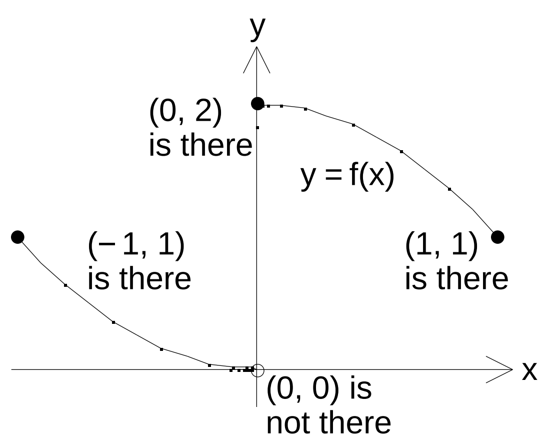
For Exercise 12



For Exercise 16

20. $f(x) = \begin{cases} x^2 & \text{if } -1 \le x < 0 \\ 2 - x^2 & \text{if } 0 \le x \le 1 \end{cases}$

Note $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$, but $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2 - x^2) = 2$, so f is discontinuous hence not differentiable at 0. $f'(x) = 0$ **nowhere**. $f(-1) = 1$, $f(0) = 2$, and $f(1) = 1$, so the absolute maximum is $f(0) = 2$ but there is no absolute minimum since although $f(x) > 0$ for all $x \quad [-1, 1]$, $\lim_{x \to 0^-} f(x) = 0$. See graph to the right.



For Exercise 20

26. $f(t) = t^3 + 6t^2 + 3t - 1$.

$f'(t) = 3t^2 + 12t + 3 = 3(t^2 + 4t + 1) = 3\left(t + 2 + \sqrt{3}\right)\left(t + 2 - \sqrt{3}\right)$.

The critical numbers are $-2 - \sqrt{3}$ and $-2 + \sqrt{3}$.

34. $f(z) = \dfrac{z + 1}{z^2 + z + 1}$, so $f'(z) = \dfrac{(z^2 + z + 1)(1) - (z + 1)(2z + 1)}{(z^2 + z + 1)^2} = \dfrac{-z^2 - 2z}{(z^2 + z + 1)^2} = -\dfrac{z(z + 2)}{(z^2 + z + 1)^2}$.
The critical numbers are 0 and -2.

46. $f(x) = 18x + 15x^2 - 4x^3$ on $[-3, 4]$.
$f'(x) = 18 + 30x - 12x^2 = -6(2x^2 - 5x - 3) = -6(2x + 1)(x - 3)$ on $(-3, 4)$.
$f(-3) = 189$, $f(-1/2) = -4.75$, $f(3) = 81$, and $f(4) = 56$.
The absolute maximum is $f(-3) = 189$ and the absolute minimum is $f(-1/2) = -4.75$.

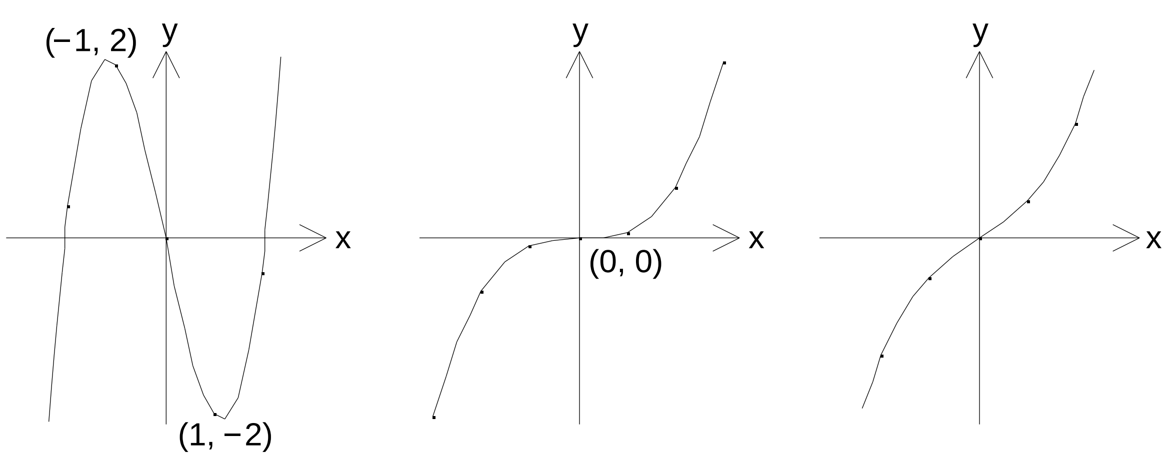
68. $g(x) = 2 + (x - 5)^3$, so $g'(x) = 3(x - 5)^2$ and g has a critical number at 5. Since $g(x) < 2$ if $x < 5$, $g(5) = 2$, and $g(x) > 2$ if $x > 5$, there is no local extremum at 5.

72. (a) If $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$.
Since $f(-\sqrt{3}) = 0$, $f(-1) = 2$, $f(0) = 0$, $f(1) = -2$, and $f(\sqrt{3}) = 0$, there are two critical numbers, -1 and 1, with a local maximum $f(-1) = 2$ and a local minimum $f(1) = -2$.

If $g(x) = x^3$, $g'(x) = 3x^2$. Since $g(x) < 0$ if $x < 0$, $g(0) = 0$, and $g(x) > 0$ if $x > 0$, there is a single critical number, 0, but $g(0) = 0$ is neither a local minimum nor a local maximum.

If $h(x) = x^3 + 3x$, $h'(x) = 3x^2 + 3 = 3(x^2 + 1)$ and h has no critical numbers. The graph of $y = h(x)$ always rises as we move to the right, and there are no local extrema.

See graphs below.



$y = f(x) = x^3 - 3x$ $\qquad$ $y = g(x) = x^3$ $\qquad$ $y = h(x) = x^3 + 3x$

(b) We cannot have more than two local extrema for a cubic polynomial function since the derivative of a cubic polynomial is a quadratic polynomial and can have at most two real roots. We have seen a case where there **are** two distinct local extrema. Although there **can** be just one critical number, in such cases the cubic either rises steadily or falls steadily, leveling out for a moment at the critical number. So there can't be just one local extremum. There certainly can be none, as our last two examples show!