MATHEMATICS 151

Assignment 11, due Monday 26 July 1999

Section 4.2 (pg. 266):

4. $f(x) = \sin x + \cos x$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$, and $f(0) = 1 = f(2\pi)$. $f'(c) = \cos c - \sin c$, so $f'(c) = 0$ when $\sin c = \cos c$, or $\tan c = 1$, at $c = \pi/4$ and at $c = 5\pi/4$.

6. If $f(x) = (x-1)^{-2}$, $f'(x) = -2(x-1)^{-3}$ except at $x = 1$. $f(0) = 1 = f(2)$.
Now $f'(c) = 0$ means $-2(c-1)^{-3} = 0$, and this has no solution.
Rolle's Theorem is not contradicted. It doesn't apply because f is discontinuous (and not differentiable) at 1, between 0 and 2.

14. $f(x) = \sqrt{x}$ on $[1, 4]$ is continuous there. $f'(x) = \frac{1}{2\sqrt{x}}$ on $(1, 4)$.
$\frac{f(4) - f(1)}{4-1} = \frac{\sqrt{4} - \sqrt{1}}{4-1} = \frac{1}{3}$. If $f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{3}$ then $2\sqrt{c} = 3$, and $c = \frac{9}{4}$.

22. Let $f(x) = x^4 + 4x + c$.

$$f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x+1)(x^2 - x + 1) = 4(x+1)\left[\left(x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right]$$

has exactly one real zero, -1.
$f(x)$ can't have three distinct real roots $p < q < r$; if it did there would be **two** values c_1 and c_2 with $p < c_1 < q < c_2 < r$ where $f'(c_1) = 0 = f'(c_2)$, by Rolle's Theorem.
So $f(x) = 0$ has at most two real roots.
If $c < 3$, it has two; if $c = 3$, it has one; if $c > 3$, it has none.
To see why, write $x^4 + 4x + c = x^4 + 4x + 3 + (c-3) = (x+1)^2\left[(x-1)^2 + 2\right] + (c-3)$.
Solving $x^4 + 4x + c = 0$ is the same as solving $(x+1)^2\left[(x-1)^2 + 2\right] = 3 - c$.
If $c < 3$, $3 - c > 0$. The first factor $(x+1)^2$ is zero when $x = -1$ and grows as large as we please as x moves away from -1 in either direction. The second factor $\left[(x-1)^2 + 2\right]$ is always at least 2. So when $3 - c > 0$, there will be two values of x, one on either side of -1, where $(x+1)^2\left[(x-1)^2 + 2\right] = 3 - c$.
However if $c = 3$, we have to solve $(x+1)^2\left[(x-1)^2 + 2\right] = 0$, and that can only occur when $x = -1$.
If $c > 3$ so that $3 - c < 0$, $(x+1)^2\left[(x-1)^2 + 2\right] = 3 - c$ can never happen.

26. $\frac{f(5) - f(2)}{5-2} = f'(c)$ for some c $(2, 5)$ by the Mean Value Theorem.
Thus $f(5) - f(2) = 3f'(c)$. Since $1 \le f'(c) \le 4$, $3 \le f(5) - f(2) \le 12$.

32. If $f'(x) = c$ and if $g(x) = cx$ (so $g'(x) = c$), then $f'(x) = g'(x)$ on any interval (a, b).
Thus $f - g$ is some constant d on (a, b), and $f(x) = g(x) + d = cx + d$ on (a, b).
Since (a, b) can be any interval, and you can catch any real number x inside some interval, $f(x) = cx + d$ on $(-\infty, +\infty)$.

36. Let $v(t)$ be the velocity (measured in mi/h) at time t (measured in hours).

Then $\frac{v(13/6) - v(2)}{13/6 - 2} = \frac{50 - 30}{1/6} = 120$.

By the Mean Value Theorem $a(c) = v'(c) = 120$ for some c, $2 < c < 13/6$.

So some time between 2:00 P.M. and 2:10 P.M. the acceleration was exactly 120 mi/h^2.

Section 4.3 (pg. 272):

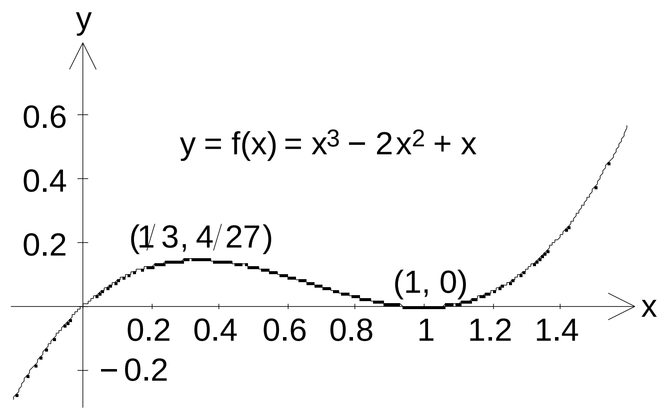
6. $f(x) = x^3 - 2x^2 + x$, so
$f'(x) = 3x^2 - 4x + 1 = (3x - 1)(x - 1)$.

$f'(x) < 0$ on $(1/3, 1)$, $f'(x) > 0$
$f'(x) > 0$ on $(-\infty, 1/3)$ $(1, +\infty)$,
and $f'(1/3) = f'(1) = 0$.

f is increasing on $(-\infty, 1/3]$ $[1, +\infty)$, decreasing on $[1/3, 1]$, has local maximum $f(1/3) = 4/27$, and has local minimum $f(1) = 0$.

$\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$,
so there are no absolute extrema.

See graph to the right.



For Exercise 6

14. $f(x) = x^{2/3}(x - 2)^2$, so $f'(x) = \frac{2}{3}x^{-1/3}(x - 2)^2 + 2x^{2/3}(x - 2) =$

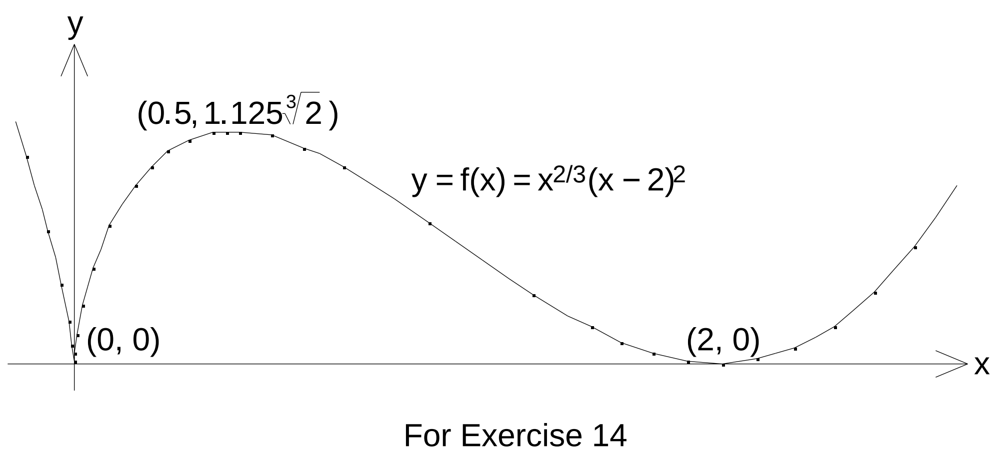
$= \frac{2}{3}x^{-1/3}(x - 2)[(x - 2) + 3x] = \frac{2}{3}x^{-1/3}(x - 2)(4x - 2) = \frac{4}{3}x^{-1/3}(x - 2)(2x - 1)$.

$f'(x) < 0$ on $(-\infty, 0)$ $(0.5, 2)$, $f'(x) > 0$ on $(0, 0.5)$ $(2, +\infty)$, $f'(0.5) = f'(2) = 0$;
$f'(0)$ does not exist.

f is increasing on $[0, 0.5]$ $[2, +\infty)$, decreasing on $(-\infty, 0]$ $[0.5, 2]$, has absolute minima $f(0) = f(2) = 0$, and has local maximum $f(0.5) = 1.125\sqrt[3]{2}$.

Since $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$, there is no absolute maximum.

See graph below.



For Exercise 14

22. $f(x) = x^5 + 4x^3 - 6$, so
$f'(x) = 5x^4 + 12x^2 = x^2(5x^2 + 12)$.

Thus $f'(x) > 0$ everywhere except at $x = 0$, and f is increasing on $(-\infty, +\infty)$ (though it levels off for a moment at the origin).

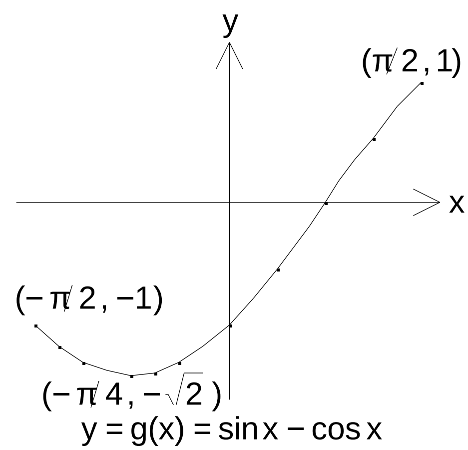
32. $g(x) = \sin x - \cos x$, $-\pi/2 \le x \le \pi/2$.

$g'(x) = \cos x + \sin x$, so $g'(x) < 0$ on $(-\pi/2, -\pi/4)$ and $g'(x) > 0$ on $(-\pi/4, \pi/2)$.

$g(-\pi/2) = -1$, $g(-\pi/4) = -\sqrt{2}$, and $g(\pi/2) = 1$,

so $g(-\pi/4) = -\sqrt{2}$ is an absolute minimum and $g(\pi/2) = 1$ is an absolute maximum.

See graph to the right.



For Exercise 32

40. Let $f(x) = \tan x - x$, $0 \le x < \pi/2$. Then $f'(x) = \sec^2 x - 1 = \tan^2 x$,
$f'(x) > 0$ on $(0, \pi/2)$, and $f(x)$ is increasing on $[0, \pi/2)$.
Since $f(0) = 0$, $f(x) > 0$ on $(0, \pi/2)$, and thus $\tan x > x$ on $(0, \pi/2)$.

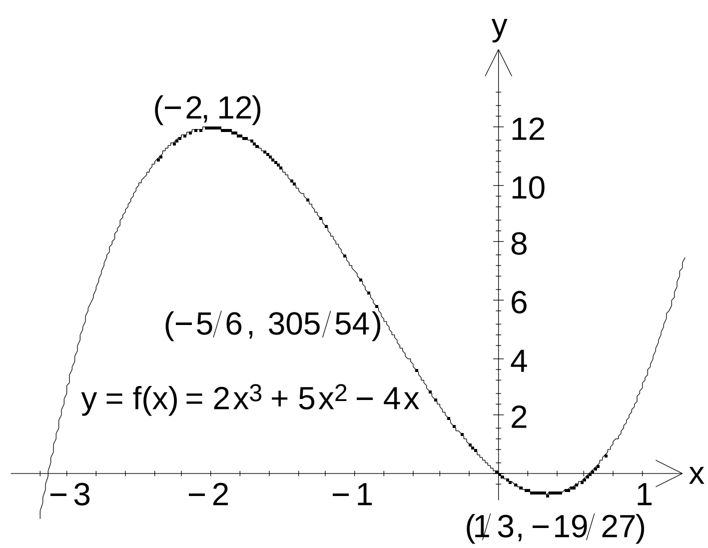
Section 4.4 (pg. 277):

4. $f(x) = 2x^3 + 5x^2 - 4x$,
so $f'(x) = 6x^2 + 10x - 4 =$
$= 2(3x^2 + 5x - 2) = 2(3x - 1)(x + 2)$
and $f''(x) = 12x + 10 = 2(6x + 5)$.

$f'(x) < 0$ on $(-2, 1/3)$,
$f'(x) > 0$ on $(-\infty, -2)$ $(1/3, +\infty)$,
and $f'(-2) = f'(1/3) = 0$.

f increases on $(-\infty, -2]$ $[1/3, +\infty)$, decreases on $[-2, 1/3]$, has local maximum $f(-2) = 12$, and has local minimum $f(1/3) = -19/27$.

$\lim_{x \to -\infty} f(x) = -\infty$, while
$\lim_{x \to +\infty} f(x) = +\infty$, and thus
there are no absolute extrema.



For Exercise 4

$f''(x) < 0$ on $(-\infty, -5/6)$,
$f''(x) > 0$ on $(-5/6, +\infty)$,
and $f''(-5/6) = 0$. The graph of $y = f(x)$ is concave downward in $(-\infty, -5/6)$ and concave upward in $(-5/6, +\infty)$, and there is an inflection point at $(-5/6, 305/54)$.

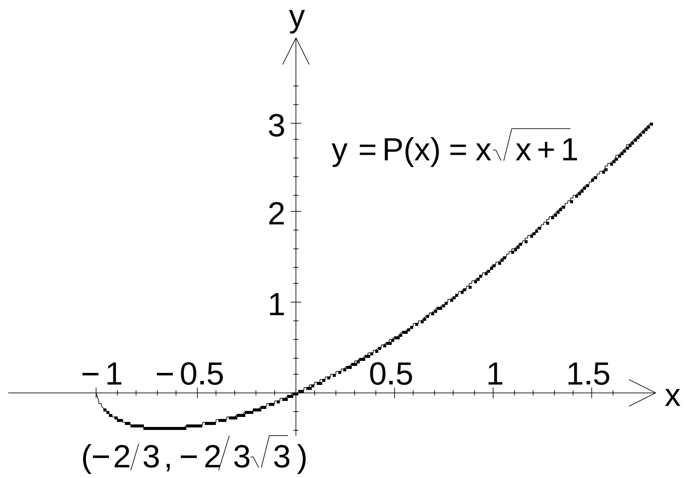
See graph above and to the right.

10. $P(x) = x\sqrt{x+1}$ on $[-1, +\infty)$.

$P'(x) = \frac{3x+2}{2(x+1)^{1/2}}$ and $P''(x) = \frac{3x+4}{4(x+1)^{3/2}}$ on $(-1, +\infty)$, so $P'(x) < 0$ on $(-1, -2/3)$, $P'(x) > 0$ on $(-2/3, +\infty)$, and $P'(-2/3) = 0$.

P decreases on $[-1, -2/3)]$, increases on $[-2/3, +\infty)$, and has an absolute minimum $P(-2/3) = -\frac{2}{3\sqrt{3}}$.

$\lim_{x \to +\infty} P(x) = +\infty$, and thus there is no absolute maximum. There are no other local extrema.



For Exercise 10

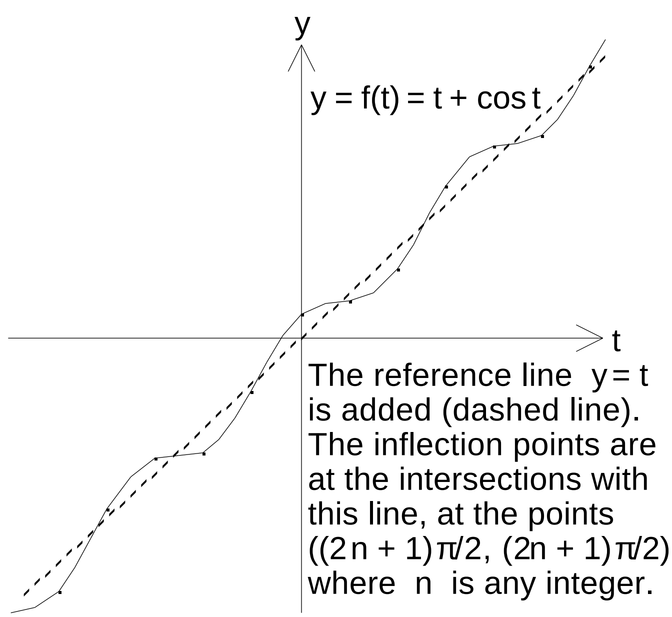
$P''(x) > 0$ on $(-1, +\infty)$, so the graph of $y = P(x)$ is concave upward everywhere and there are no inflection points.

See graph above and to the right.

14. $f(t) = t + \cos t$, so $f'(t) = 1 - \sin t$. Then $f'(t) > 0$ except when t is of the form $\frac{4k+1}{2}\pi$, k an integer.

Thus f is increasing everywhere, leveling off momentarily at each value $\frac{4k+1}{2}\pi$. There are no local extrema.

$f''(t) = -\cos t$, so $f''(t) > 0$ on each interval $\left(\frac{4k+1}{2}\pi, \frac{4k+3}{2}\pi\right)$ where the graph of $y = f(t)$ is concave upward, and $f''(t) < 0$ on each interval $\left(\frac{4k-1}{2}\pi, \frac{4k+1}{2}\pi\right)$ where the graph of $y = f(t)$ is concave downward. The inflection points are the points $\left(\frac{2n+1}{2}\pi, \frac{2n+1}{2}\pi\right)$.



For Exercise 14

See graph to the right.

38. If $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 12x^2$ so $f''(0) = 0$. But $f''(x) > 0$ in $(-\infty, 0)\quad(0, +\infty)$ so the graph of $y = f(x)$ is concave upward in $(-\infty, 0)\quad(0, +\infty)$. Since the concavity does not change there, the point $(0, 0)$ is not an inflection point.

42. If $g(x) = (f(x))^2$, $g'(x) = 2f(x)f'(x)$ and $g''(x) = 2(f'(x))^2 + 2f(x)f''(x) \geq 2f(x)f''(x) > 0$ on I if $f(x) > 0$ and $f''(x) > 0$ on I.