MATHEMATICS 151

Assignment 12, due Wednesday 28 July 1999

Section 4.5 (pg. 287):

8. $y = \dfrac{x}{x^2 - 9} = \dfrac{x}{(x+3)(x-3)}$.

Domain: $(-\infty, -3) \quad (-3, 3) \quad (3, +\infty)$.

Intercepts: The x-intercept and the y-intercept are both 0.

Symmetry: $\dfrac{(-x)}{(-x)^2 - 9} = -\dfrac{x}{x^2 - 9}$ so the graph is symmetric about the origin.

Asymptotes: The vertical asymptotes are $x = -3$ and $x = 3$.
The horizontal asymptote is $y = 0$, approached from below as $x \to -\infty$ and approached from above as $x \to +\infty$.

Intervals of increase and decrease: $y' = \dfrac{(x^2-9)\cdot 1 - x\cdot 2x}{(x+3)^2(x-3)^2} = -\dfrac{x^2+9}{(x+3)^2(x-3)^2} < 0$ for all x in the function's domain $(-\infty, -3) \quad (-3, 3) \quad (3, +\infty)$, so the function is decreasing on $(-\infty, -3) \quad (-3, 3) \quad (3, +\infty)$.

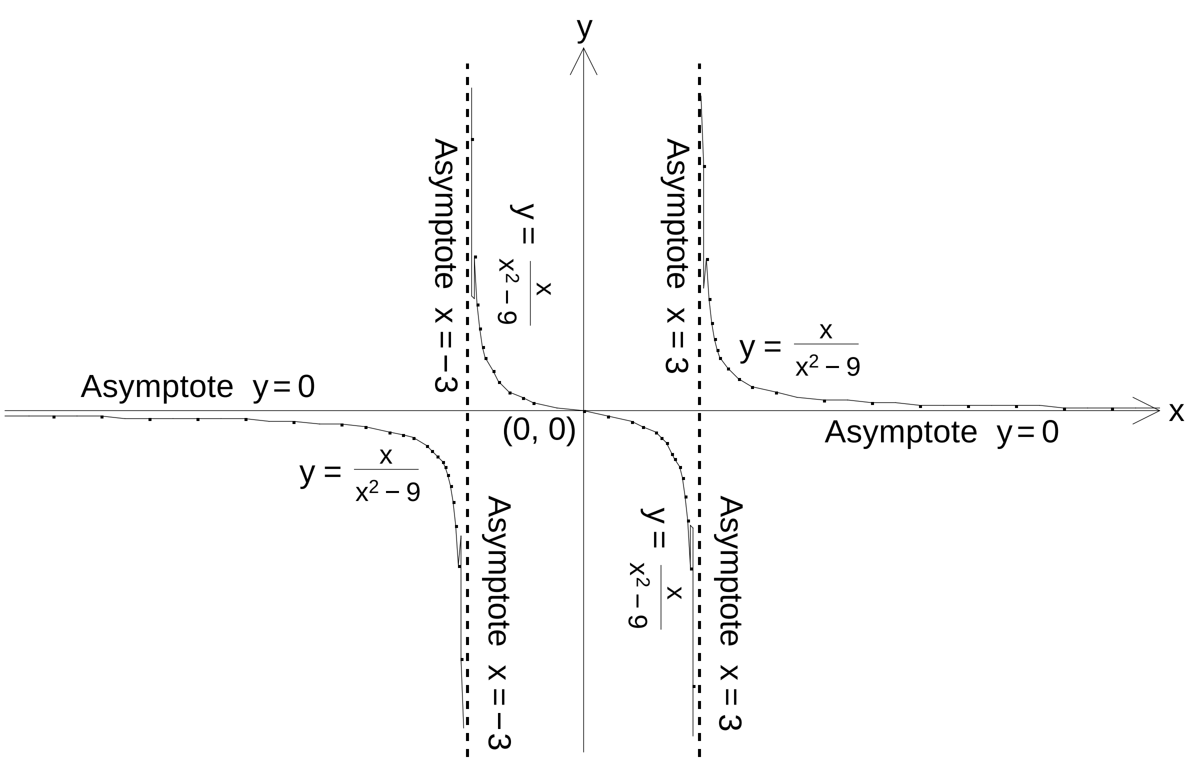
Local extrema: There are no local extrema.

Concavity: $y'' = -\dfrac{(x+3)^2(x-3)^2\cdot 2x - (x^2+9)[2(x+3)(x-3)^2 + 2(x+3)^2(x-3)]}{(x+3)^4(x-3)^4} =$

$= \dfrac{-2x(x+3)(x-3) + 2(x^2+9)(2x)}{(x+3)^3(x-3)^3} = \dfrac{2x(x^2+27)}{(x+3)^3(x-3)^3}$.

The graph is concave downward on $(-\infty, -3) \quad (0, 3)$ and concave upward on $(-3, 0) \quad (3, +\infty)$. There is an inflection point at $(0, 0)$.

See graph below.



10. $y = \dfrac{1}{x^2(x+3)}$.

Domain: $(-\infty, -3)$ $(-3, 0)$ $(0, +\infty)$.

Intercepts: There are no intercepts.

Symmetry: $\dfrac{1}{(-x)^2\{(-x)+3\}} \neq \pm \dfrac{1}{x^2(x+3)}$, so there is no symmetry.

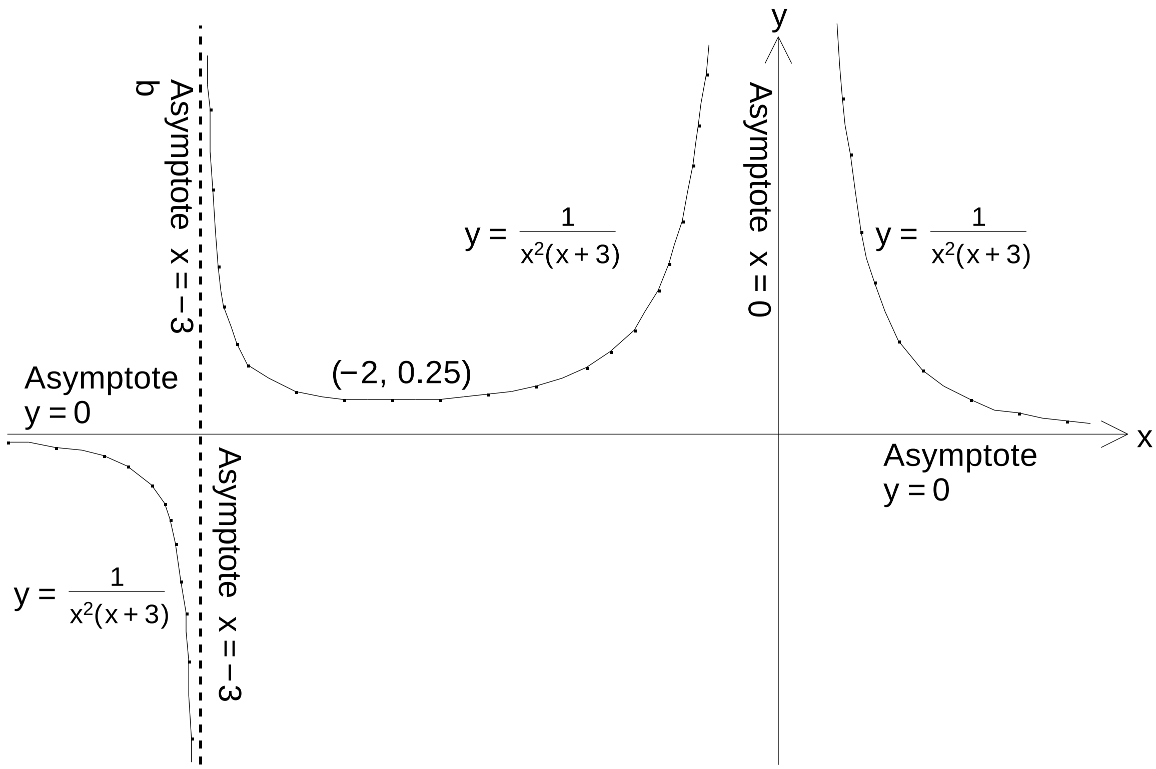
Asymptotes: The vertical asymptotes are $x = 0$ and $x = -3$.
The horizontal asymptote is $y = 0$, approached from below as $x \to -\infty$ and approached from above as $x \to +\infty$.

Intervals of increase and decrease: $y' = -\dfrac{3(x+2)}{x^3(x+3)^2}$. The function is increasing on $[-2, 0)$ and decreasing on $(-\infty, -3)$ $(-3, -2]$ $(0, +\infty)$.

Local extrema: There is a local minimum $f(-2) = \dfrac{1}{4}$.

Concavity: $y'' = \dfrac{6(2x^2 + 8x + 9)}{x^4(x+3)^3} = \dfrac{6[2(x+2)^2 + 1]}{x^4(x+3)^3}$.
The graph is concave downward on $(-\infty, -3)$ and concave upward on $(-3, 0)$ $(0, +\infty)$. There are no inflection points.

See graph below.



14. $y = \dfrac{1 - x^2}{x^3}$.

Domain: $(-\infty, 0) \quad (0, +\infty)$.

Intercepts: The x-intercepts are $x = \pm 1$. There is no y-intercept.

Symmetry: $\dfrac{1 - (-x)^2}{(-x)^3} = -\dfrac{1 - x^2}{x^3}$, so the graph is symmetric with respect to the origin.

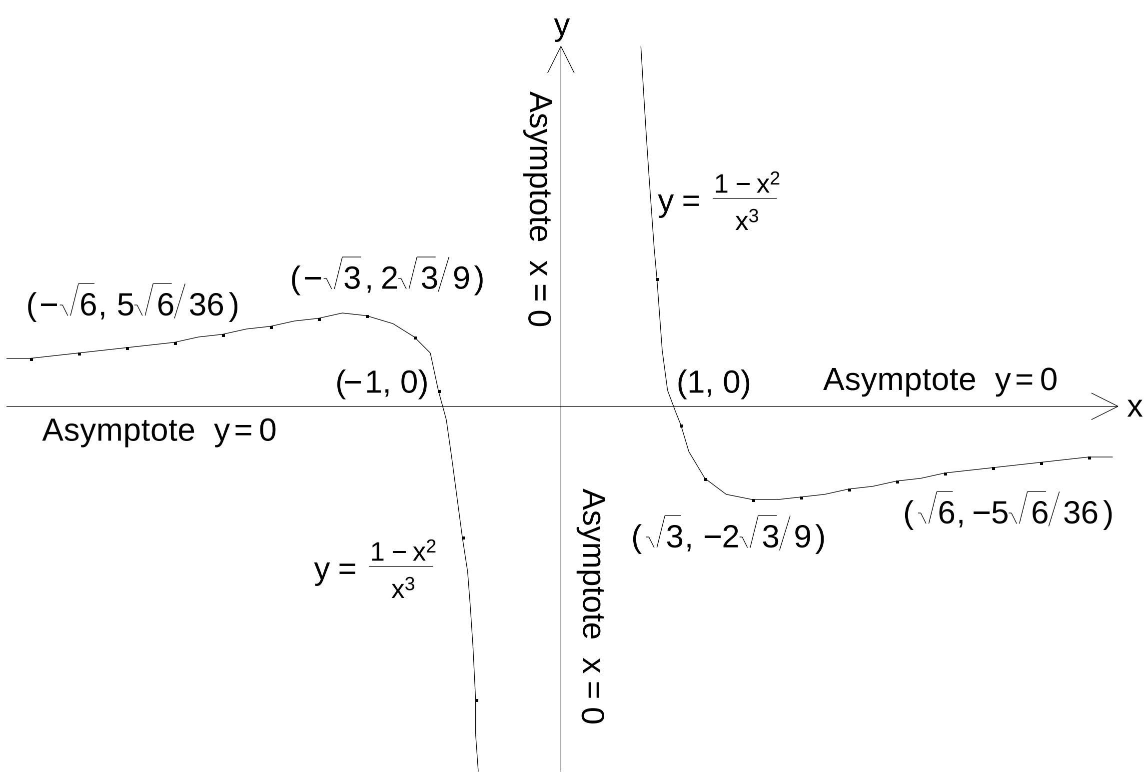
Asymptotes: The vertical asymptote is $x = 0$.
The horizontal asymptote is $y = 0$, approached from above as $x \to -\infty$ and approached from below as $x \to +\infty$.

Intervals of increase and decrease: $y' = \dfrac{x^2 - 3}{x^4}$ so the function is increasing on $(-\infty, -\sqrt{3}\,] \quad [\sqrt{3}, +\infty)$ and decreasing on $[-\sqrt{3}, 0) \quad (0, \sqrt{3}\,]$.

Local extrema: There is a local maximum $f(-\sqrt{3}\,) = \dfrac{2}{3\sqrt{3}}$ and a local minimum $f(\sqrt{3}\,) = -\dfrac{2}{3\sqrt{3}}$.

Concavity: $y'' = \dfrac{2(6 - x^2)}{x^5}$. The graph is concave downward on $(-\sqrt{6}, 0) \quad (\sqrt{6}, +\infty)$ and it is concave upward on $(-\infty, -\sqrt{6}\,) \quad (0, \sqrt{6}\,)$.
The inflection points are $(-\sqrt{6},\ 5\sqrt{6}/36)$ and $(\sqrt{6},\ -5\sqrt{6}/36)$.

See graph below.



18. $y = \sqrt{\dfrac{x}{x-5}} = \sqrt{1 + \dfrac{5}{x-5}}$.

Domain: $(-\infty, 0] \quad (5, +\infty)$.

Intercepts: Both intercepts are 0.

Symmetry: There is no symmetry. To begin with, look at the domain!

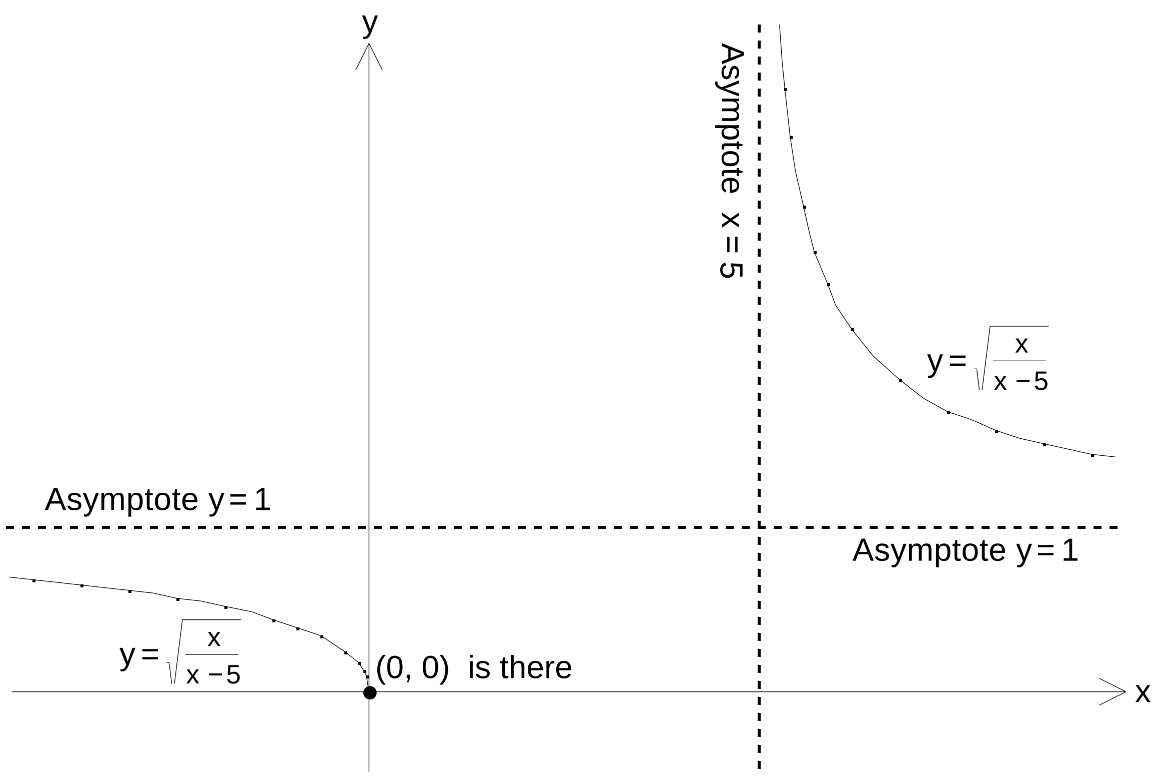
Asymptotes: There is a vertical asymptote $x = 5$, approached from the right only. The horizontal asymptote is $y = 1$, approached from below as $x \to -\infty$ and approached from above as $x \to +\infty$.

Intervals of increase and decrease: $y' = -\frac{5}{2}\left(\frac{x-5}{x}\right)^{1/2}\left(\frac{1}{x-5}\right)^{2}$ which is negative on $(-\infty, 0) \quad (5, +\infty)$, hence the function is decreasing on $(-\infty, 0] \quad (5, +\infty)$.

Local extrema: There is an absolute minimum $f(0) = 0$.

Concavity: $y'' = \frac{5}{4}(4x-5)[x^3(x-5)^5]^{-1/2}$;
the graph is concave downward on $(-\infty, 0)$ and it is concave upward on $(5, +\infty)$. There are no inflection points.

See graph below.



32. $y = 2\sin x + \sin^2 x$. This can also be written as $y = 2\cos\left(x - \frac{\pi}{2}\right) + \cos^2\left(x - \frac{\pi}{2}\right)$ or as $y = -2\cos\left(x - \frac{3\pi}{2}\right) + \cos^2\left(x - \frac{3\pi}{2}\right)$.

Domain: $(-\infty, +\infty)$.

Intercepts: The x-intercepts are all numbers of the form $n\pi$, n an integer.
The y-intercept is 0.

Symmetry: $\sin(-x)(2 + \sin(-x)) = -\sin x(2 - \sin x) \neq \pm[\sin x(2 + \sin x)]$ so there is no symmetry in the usual sense about either axis or the origin.
Since the cosine function is even, there is symmetry with respect to any vertical line of the form $x = \pi/2 + 2n\pi$ or $x = 3\pi/2 + 2n\pi$, n any integer; these can be put together as $x = \pi/2 + k\pi$, k any integer.

The function is periodic, with period 2π.

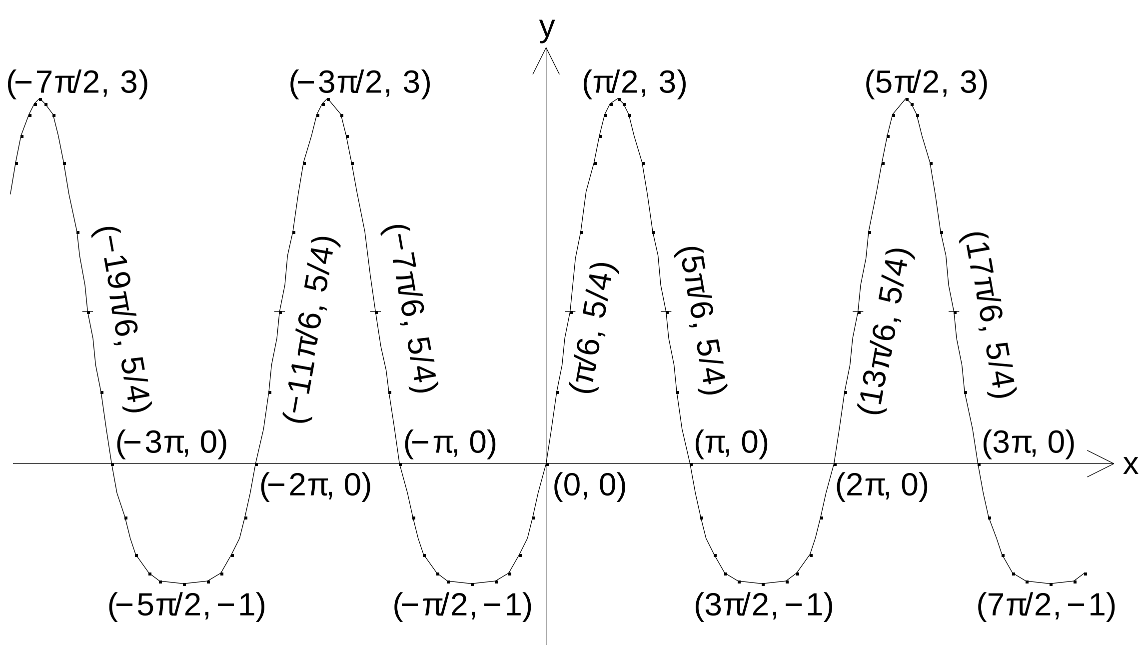
Asymptotes: There are no asymptotes.

Intervals of increase and decrease: $y' = 2\cos x + 2\sin x\cos x = 2\cos x(1 + \sin x)$.
The function is increasing on all intervals of the form $[-\pi/2 + 2n\pi, \pi/2 + 2n\pi]$ and decreasing on all intervals of the form $[\pi/2 + 2n\pi, 3\pi/2 + 2n\pi]$, n an integer.

Local extrema: There are absolute maxima $f(\pi/2 + 2n\pi) = 3$ and absolute minima $f(-\pi/2 + 2n\pi) = -1$, n an integer.

Concavity: $y'' = -2\sin x + 2\cos^2 x - 2\sin^2 x = -2\sin x + 2 - 4\sin^2 x =$
$= -2(2\sin^2 x + \sin x - 1) = -2(2\sin x - 1)(\sin x + 1)$.
The graph is concave downward on the intervals $(\pi/6 + 2n\pi, 5\pi/6 + 2n\pi)$ and concave upward on the intervals $(-7\pi/6 + 2n\pi, \pi/6 + 2n\pi)$.
The points $(\pi/6 + 2n\pi, 5/4)$ and $(5\pi/6 + 2n\pi, 5/4)$ are inflection points.
Here again n can be any integer.

See graph below.



$y = 2\sin x + \sin^2 x$

60. $y = \dfrac{x^2}{2x+5}$.

Domain: $(-\infty, -2.5) \quad (-2.5, +\infty)$.

Intercepts: The x-intercept and the y-intercept are both 0.

Symmetry: There is no symmetry in the usual sense about either axis or the origin since $\dfrac{(-x)^2}{2(-x)+5} \neq \pm \dfrac{x^2}{2x+5}$.

The graph is however symmetric about the point $(-2.5, -2.5)$. To see this, write

$y = \dfrac{x^2}{2x+5} = \dfrac{(x+2.5)^2 - 5(x+2.5) + 2.5^2}{2(x+2.5)}$, so $y + 2.5 = \frac{1}{2}(x+2.5) + 3.125(x+2.5)^{-1}$.

If in this expression for $y + 2.5$ we replace $x + 2.5$ by $-(x+2.5)$, we obtain a correct expression for $-(y + 2.5)$.

Asymptotes: The vertical asymptote is $x = -2.5$.

Since $y = \frac{1}{2}x - 1.25 + \dfrac{6.25}{2x+5}$, there is a slant asymptote $y = \frac{1}{2}x - 1.25 = \frac{1}{2}(x - 2.5)$ approached from below as $x \to -\infty$ and approached from above as $x \to +\infty$.

Intervals of increase and decrease: $y' = \dfrac{(2x+5)(2x) - x^2 \cdot 2}{(2x+5)^2} = \dfrac{2x^2+10x}{(2x+5)^2} = \dfrac{2x(x+5)}{(2x+5)^2}$

The function is increasing on $(-\infty, -5] \quad [0, +\infty)$ and decreasing on $[-5, -2.5) \quad (-2.5, 0]$.

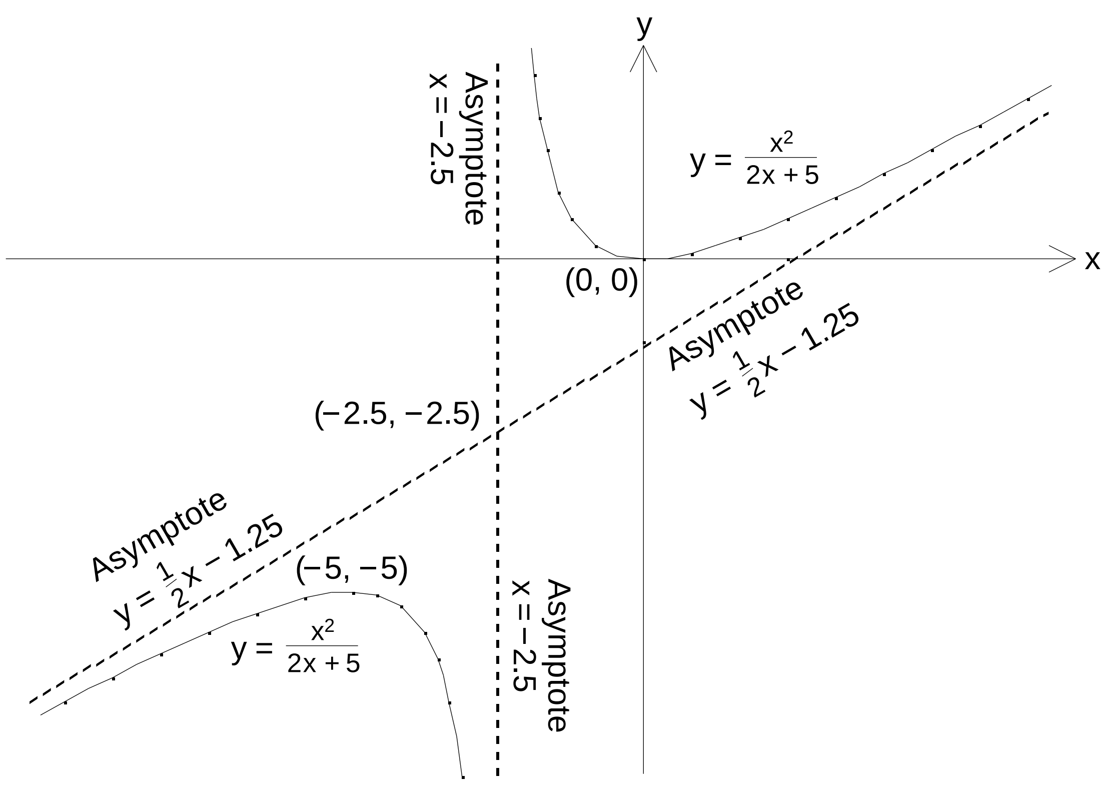
Local extrema: There is a local maximum $f(-5) = -5$ and a local minimum $f(0) = 0$.

Concavity: $y'' = \dfrac{(2x+5)^2[2(x+5)+2x] - 2x(x+5)\cdot 2\cdot 2\cdot (2x+5)}{(2x+5)^4} =$

$= \dfrac{(2x+5)(4x+10) - 8x(x+5)}{(2x+5)^3} = \dfrac{2(2x+5)^2 - 8x(x+5)}{(2x+5)^3} = \dfrac{50}{(2x+5)^3}$.

The graph is concave downward on $(-\infty, -2.5)$ and concave upward on $(-2.5, +\infty)$. There are no inflection points.

See graph below.

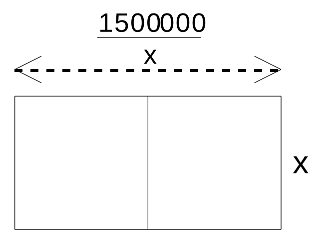


Section 4.7 (pg. 299):

2. If one number is x and the other number is $x + 100$ then their product is $P(x) = x(x + 100) = x^2 + 100x$ which we wish to minimize. Here $-\infty < x < +\infty$.
$P'(x) = 2x + 100 = 2(x + 50)$ so $P(x)$ decreases in $(-\infty, -50]$, increases in $[-50, +\infty)$, and has absolute minimum $P(-50) = -2500$.
Checking, one number is -50, the other is 50, their difference is 100, and their product is -2500.

We can avoid calculus by writing $P(x) = (x + 50)^2 - 2500$, which makes it obvious that the way to minimize $P(x)$ is to take $x = -50$ so that $(x + 50)^2 = 0$.

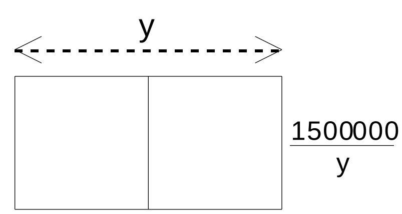
6. Let x be the width of the field, in feet.



The field length is $\dfrac{1500000}{x}$.
The total fence length (which the farmer wishes to minimize) is $f(x) = \dfrac{3000000}{x} + 3x$. Here $0 < x < +\infty$.
Since $f'(x) = -\dfrac{3000000}{x^2} + 3 = \dfrac{3}{x^2}(x^2 - 1000000) =$
$= \dfrac{3}{x^2}(x + 1000)(x - 1000)$, $f(x)$ decreases on $(0, 1000]$, and increases on $[1000, +\infty)$.

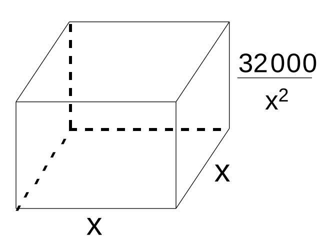
The minimum fence length occurs when the field width is 1000 ft, the field length is 1500 ft, and the total fence length is 6000 ft. Notice that the same total amount of fencing (3000 ft) is used in the width direction as is used in the length direction.

Alternatively let y be the length of the field, in feet.



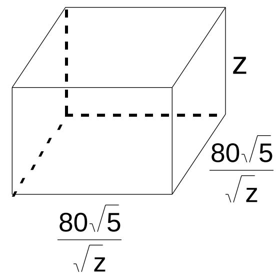
The field width is $\dfrac{1500000}{y}$.
Total fence length is $g(y) = \dfrac{4500000}{y} + 2y$, $0 < y < +\infty$.
To minimize this, since $g'(y) = -\dfrac{4500000}{y^2} + 2 =$
$= \dfrac{2}{y^2}(y^2 - 2250000) = \dfrac{2}{y^2}(y + 1500)(y - 1500)$ the farmer should make the fence length y be 1500 ft and the width be 1000 ft, as before.

8. Let x be the base width, in centimetres.



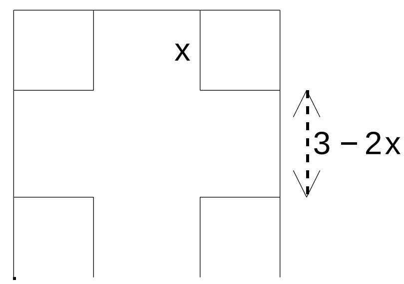
The height is $\dfrac{32000}{x^2}$.
$S(x) = x^2 + 4 \cdot x \cdot \dfrac{32000}{x^2} = x^2 + \dfrac{128000}{x}$ is the surface area.
Here $0 < x < +\infty$.
$S'(x) = 2x - \dfrac{128000}{x^2} = \dfrac{2}{x^2}(x^3 - 64000)$, so $S(x)$ decreases in $(0, 40]$ and increases in $[40, +\infty)$, and the surface area is minimized (at $4800\ cm^2$) when the box is 40 cm wide and long, and 20 cm high.
Notice that the front and back of the box together will then have an area of $1600\ cm^2$, the same as that of the right and left sides, and the same as that of the base.

Alternatively let z be the box height, in centimetres.
The base area must be $\frac{32000}{z}$ and the square base must have side length $\frac{80\sqrt{5}}{\sqrt{z}}$.



$T(z) = \frac{32000}{z} + 4\ \frac{80\sqrt{5}}{\sqrt{z}}\ z = \frac{32000}{z} + 320\sqrt{5}\sqrt{z}$ is the surface area.
Here $0 < z < +\infty$.
$T'(z) = -\frac{32000}{z^2} + \frac{160\sqrt{5}}{\sqrt{z}} = \frac{160\sqrt{5}}{z^2}\left(z^{3/2} - 40\sqrt{5}\right)$, so $T(z)$ is minimized when the height z is 20 cm and the base is 40 cm on a side, as before.

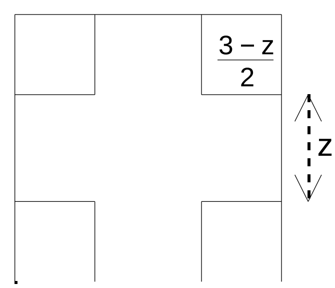
12. Let each of the four squares removed from the corners have side length x, measured in feet.
The box base side length will be $3 - 2x$.
Its height will be x.
Its volume will be $V(x) = x(3 - 2x)^2 = 4x^3 - 12x^2 + 9x$.
Here $0 < x < 1.5$.
Then $V'(x) = 12x^2 - 24x + 9 = 3(4x^2 - 8x + 3) =$
$= 3(2x - 1)(2x - 3)$ so the volume increases in $(0, 0.5]$ and decreases in $[0.5, 1.5)$.



The maximum volume, 2 ft^3, occurs when the cut-out squares are 0.5 ft on a side.

If we work in inches instead of in feet, letting y be the side length of each cut-out square the volume will be $W(y) = y(36 - 2y)^2 = 4y^3 - 144y^2 + 1296y$, $0 < y < 18$.
Then $W'(y) = 12y^2 - 288y + 1296 = 12(y^2 - 24y + 108) = 12(y - 6)(y - 18)$.
The maximum volume of 3456 in^3 occurs when the removed squares are each 6 in on a side, as before.

For an alternative approach, let z be the width of the square base, in feet, after the four corner squares are removed and the sides bent upwards.
The height will be $\frac{3 - z}{2}$.



$A(z) = z^2\left(\frac{3 - z}{2}\right) = -\frac{1}{2}z^3 + \frac{3}{2}z^2$ will be the volume.
Here $0 < z < 3$.
$A'(z) = -\frac{3}{2}z^2 + 3z = \frac{3}{2}z(-z + 2)$, so the volume is maximized at 2 ft^3 when the square box base measures 2 ft on a side (and the cut-out squares measure 0.5 ft on a side).

One can work in inches, with the square box base w inches on a side, the box height $\frac{36 - w}{2}$, and the volume $B(w) = w^2\left(\frac{36 - w}{2}\right) = 18w^2 - \frac{1}{2}w^3$, $0 < w < 36$.
Then $B'(w) = 36w - \frac{3}{2}w^2 = \frac{3}{2}w(24 - w)$, so the largest volume occurs when the square box base is 24 in on a side and the box height is 6 in.

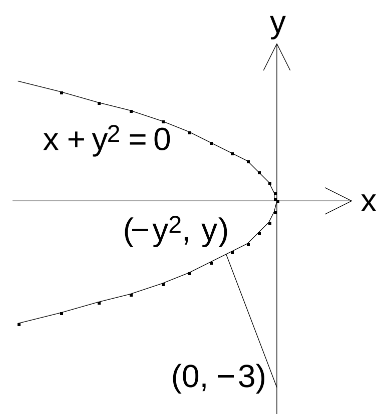
16. Let y be the second coordinate of an arbitrary point on the parabola.
The first coordinate will be $-y^2$.
The distance from the point to $(0, -3)$ will be

$$f(y) = \sqrt{(-y^2 - 0)^2 + (y - (-3))^2} = (y^4 + y^2 + 6y + 9)^{1/2}.$$

Here $-\infty < y < +\infty$.

$$f'(y) = (2y^3 + y + 3)(y^4 + y^2 + 6y + 9)^{-1/2} =$$
$$= (y + 1)(2y^2 - 2y + 3)(y^4 + y^2 + 6y + 9)^{-1/2} =$$
$$= (y + 1)\left[2\left(y - \frac{1}{2}\right)^2 + \frac{5}{2}\right](y^4 + y^2 + 6y + 9)^{-1/2},$$

so f decreases in $(-\infty, -1]$ and increases in $[-1, +\infty)$.
The closest point on the parabola to $(0, -3)$ is $(-1, -1)$.



Notice the tangent line to the parabola at $(-1, -1)$ (which has slope $1/2$) and the line segment between $(-1, -1)$ and $(0, -3)$ (which has slope -2) are perpendicular.

For an alternative solution, minimize the **square** of the distance,
$g(y) = (-y^2 - 0)^2 + (y - (-3))^2 = y^4 + y^2 + 6y + 9$, instead of the distance $f(y)$.

$$g'(y) = 4y^3 + 2y + 6 = 2(2y^3 + y + 3) =$$
$$= 2(y + 1)\left[2\left(y - \frac{1}{2}\right)^2 + \frac{5}{2}\right];$$ the rest of the solution is as before.

26. Let the rectangle width be x ft.

The semicircle at the top is of length $\frac{\pi x}{2}$.

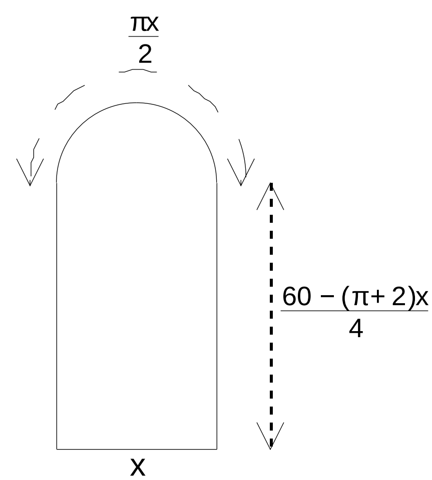
The rectangle height is $\dfrac{30 - \frac{\pi x}{2} - x}{2} = \dfrac{60 - (\pi + 2)x}{4}$.

The window area is $A(x) = \left[\dfrac{60 - (\pi + 2)x}{4}\right] x + \dfrac{1}{2}\pi\left(\dfrac{x}{2}\right)^2 =$

$= \left(-\frac{1}{8}\pi - \frac{1}{2}\right)x^2 + 15x$. Here $0 < x < \dfrac{60}{\pi + 2}$.

$A'(x) = -\left(\frac{1}{4}\pi + 1\right)x + 15$, so $A(x)$ is maximized when

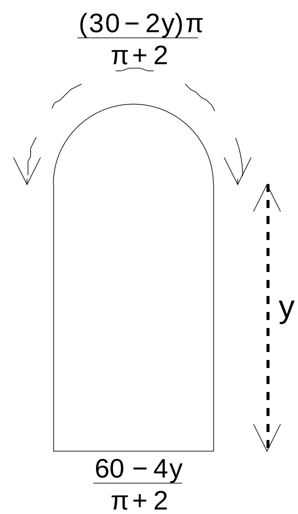
$x = \dfrac{15}{\frac{1}{4}\pi + 1} = \dfrac{60}{\pi + 4}$.



The window width should be $\dfrac{60}{\pi + 4} \approx 8.40$ ft, and the height of the rectangular part should be $\dfrac{30}{\pi + 4} \approx 4.20$ ft.

Alternatively one can let y be the rectangle height in feet, so that $30 - 2y$ ft remain to be shared between the width at the bottom and the semicircular arc at the top.

Since the semicircle arc length is $\frac{\pi}{2}$ times as long as the diameter, the rectangle width must be $\dfrac{60 - 4y}{\pi + 2}$.



The total area is $B(y) = y\left(\frac{60 - 4y}{\pi + 2}\right) + \frac{1}{2}\pi\left(\frac{30 - 2y}{\pi + 2}\right)^2 =$

$= -2\frac{\pi + 4}{(\pi + 2)^2}y^2 + \frac{120}{(\pi + 2)^2}y + \frac{450\pi}{(\pi + 2)^2} = -\frac{2}{(\pi + 2)^2}\left[(\pi + 4)y^2 - 60y - 225\pi\right],\ 0 < y < 15.$

$B'(y) = -\frac{4}{(\pi + 2)^2}\left[(\pi + 4)y - 30\right]$, so the area is maximized by making the rectangle height $\frac{30}{\pi + 4}$ ft, as before.

For a third way, once could let the semicircular arc length be z ft.

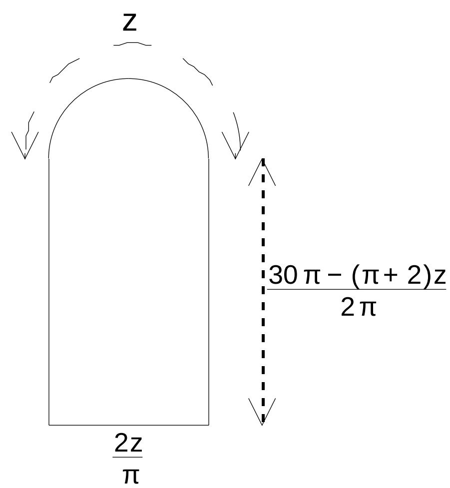
The rectangle width would then be $\frac{2z}{\pi}$, and its height would be $\frac{30 - z - \frac{2z}{\pi}}{2} = \frac{30\pi - (\pi + 2)z}{2\pi}$.

The total area would be

$C(z) = \frac{30\pi - (\pi + 2)z}{2\pi} \cdot \frac{2z}{\pi} + \frac{z^2}{2\pi} =$

$= \frac{1}{2\pi^2}\left[-(\pi + 4)z^2 + 60\pi z\right],\ 0 < z < \frac{30\pi}{\pi + 2}$.

$C'(z) = \frac{1}{\pi^2}\left[-(\pi + 4)z + 30\pi\right]$, so the total area $C(z)$ is maximized when the semicircular arc length z is $\frac{30\pi}{\pi + 4}$ ft and the rectangle is $\frac{60}{\pi + 4}$ ft wide and $\frac{30}{\pi + 4}$ ft high, the same solution.

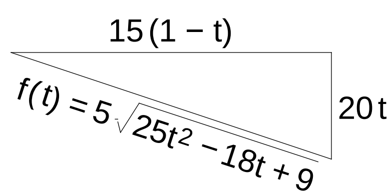


36. Measure time t in hours after 2:00 P.M.
The first boat is $20t$ km south of the pier and the second boat is $15(1 - t)$ km west of the pier at time t.
The distance in kilometres between the boats is

$f(t) = \sqrt{(20t)^2 + [15(1 - t)]^2} = 5\sqrt{(4t)^2 + 9(1 - t)^2} =$

$= 5\sqrt{25t^2 - 18t + 9}$ for $0 \le t \le 1$.

$f'(t) = \frac{5(25t - 9)}{\sqrt{25t^2 - 18t + 9}}$, so $f(t)$ is decreasing for $0 \le t \le \frac{9}{25}$ and increasing for $\frac{9}{25} \le t \le 1$.

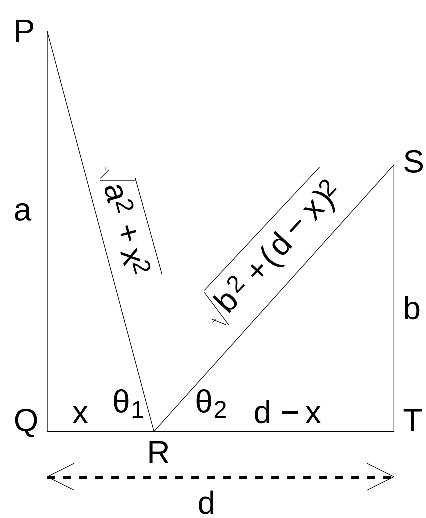


The boats were closest together when $t = \frac{9}{25}$, or at 2:21:36 P.M.

At that time they were 12 km apart.

It is easier to work with the square of the distance (it does not involve a square root).
Let $g(t) = (f(t))^2 = (20t)^2 + [15(1 - t)]^2 =$
$= 25[(4t)^2 + 9(1 - t)^2] = 25(25t^2 - 18t + 9)$.
Then $g'(t) = 25(50t - 18)$; the rest of the solution is as before.

44. Let a be the height of the pole PQ, b the height of the pole ST, d the separation between Q and T, and x the distance from Q to R, all measured in the same units.
The total rope length, which is to be minimized, is

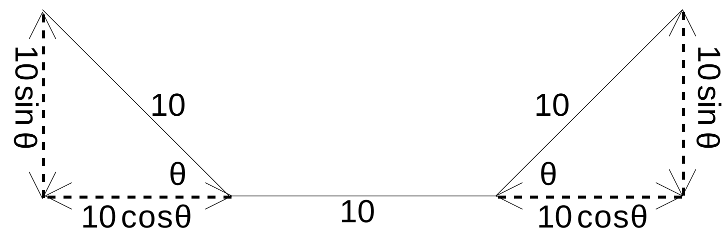


$f(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2} = \sqrt{a^2 + x^2} + \sqrt{b^2 + d^2 - 2dx + x^2}$. Here $0 < x < d$.

$$f'(x) = \frac{x}{\sqrt{a^2 + x^2}} + \frac{x - d}{\sqrt{b^2 + d^2 - 2dx + x^2}} = \cos\theta_1 - \cos\theta_2 .$$

As x increases, θ_1 decreases and θ_2 increases, so $f'(x)$ increases from negative values through to positive values. Thus $f(x)$ decreases until x becomes large enough to make $\cos\theta_1 = \cos\theta_2$, and thereafter increases; it is minimized when $\theta_1 = \theta_2$.

48. In the figure,
$10 \sin\theta$ is the gutter height.
The top width is $10(1 + 2\cos\theta)$
and the bottom width is 10,
so the average width is
$10(1 + \cos\theta)$.
The cross-section area is thus



$A(\theta) = 100(1 + \cos\theta)(\sin\theta) = 100(\sin\theta + \cos\theta\sin\theta)$, where $0 < \theta < \frac{2\pi}{3}$.

(If $\theta = 0$ the gutter is flat; if $\theta = \frac{2\pi}{3}$ the gutter closes up and becomes a triangle.)

$A'(\theta) = 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(2\cos^2\theta + \cos\theta - 1) =$
$= 100(2\cos\theta - 1)(\cos\theta + 1)$.

Thus $A(\theta)$ increases in $(0, \pi/3]$, decreases in $[\pi/3, 2\pi/3)$, and is maximized as $75\sqrt{3}$ when $\theta = \pi/3$.