MATHEMATICS 151

Assignment 13, due Wednesday 04 August 1999

Section 4.9 (pg. 313):

2. If $f(x) = x^3 - 4x^2 + 17$, then the most general antiderivative for $f(x)$ is
$F(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + 17x + C$, on $(-\infty, +\infty)$.

6. If $f(x) = \sqrt[3]{x^2} - \sqrt{x^3} = x^{2/3} - x^{3/2}$, then the most general antiderivative for $f(x)$ is
$F(x) = \frac{3}{5}x^{5/3} - \frac{2}{5}x^{5/2} + C$, on $(0, +\infty)$.

10. If $f(x) = x^{2/3} + 2x^{-1/3}$, then the most general antiderivative for $f(x)$ is

$$F(x) = \begin{cases} \frac{3}{5}x^{5/3} + 3x^{2/3} + C_1 & \text{on } (-\infty, 0) \\ \frac{3}{5}x^{5/3} + 3x^{2/3} + C_2 & \text{on } (0, +\infty) \end{cases}$$

. Here C_1 and C_2 can be different constants.

12. If $f(t) = \sin t - 2\sqrt{t} = \sin t - 2t^{1/2}$, then the most general antiderivative for $f(t)$ is
$F(t) = -\cos t - \frac{4}{3}t^{3/2} + C$, on $(0, +\infty)$.

14. If $f(\theta) = \theta + \sec\theta\tan\theta$, then the most general antiderivative for $f(\theta)$ is
$F(\theta) = \frac{1}{2}\theta^2 + \sec\theta + C_n$, on any interval $(-\pi/2 + n\pi, \pi/2 + n\pi)$, n an integer.
(Different constants C_n can be used on different intervals.)

18. $f''(x) = 60x^4 - 45x^2$.
$f'(x) = 12x^5 - 15x^3 + A$, where A is an arbitrary constant, on $(-\infty, +\infty)$.
$f(x) = 2x^6 - \frac{15}{4}x^4 + Ax + B$, where B is also an arbitrary constant, on $(-\infty, +\infty)$.

24. $f'(x) = 12x^2 - 24x + 1$.
$f(x) = 4x^3 - 12x^2 + x + C$.
$f(1) = -2$, so $4 \cdot 1^3 - 12 \cdot 1^2 + 1 + C = -2$, and $C = 5$.
$f(x) = 4x^3 - 12x^2 + x + 5$.

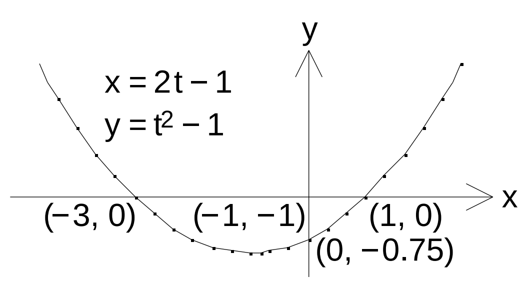
32. $f''(x) = 20x^3 - 10$.
$f'(x) = 5x^4 - 10x + A$, where A is an arbitrary constant, on $(-\infty, +\infty)$.
$f(x) = x^5 - 5x^2 + Ax + B$, where B is also an arbitrary constant, on $(-\infty, +\infty)$.

58. $a(t) = \cos t + \sin t$.
$v(t) = \sin t - \cos t + A$.
$v(0) = 5$, so $\sin 0 - \cos 0 + A = 5$, hence $0 - 1 + A = 5$, and $A = 6$.
$v(t) = \sin t - \cos t + 6$.
$s(t) = -\cos t - \sin t + 6t + B$.
$s(0) = 0$, so $-\cos 0 - \sin 0 + 6\cdot 0 + B = 0$, hence $-1 - 0 + 0 + B = 0$, and $B = 1$.
$s(t) = -\cos t - \sin t + 6t + 1$.

72. $a(t) = -40$.
$v(t) = -40t + A$.
$v(0) = -40\cdot 0 + A$, so $A = v(0)$.
$v(t) = -40t + v(0)$.
$s(t) = -20t^2 + v(0)t + B$.
$s(0) = -20\cdot 0^2 + v(0)\cdot 0 + B$, so $B = s(0)$.
$s(t) = -20t^2 + v(0)t + s(0)$.
When $v(t) = 0$, $s(t) - s(0) = 160$ from the statement about the length of the skid marks.
$-20t^2 + v(0)t = 160$ when the car stopped.
When $v(t) = 0$, $-40t + v(0) = 0$ and $t = \frac{v(0)}{40}$ when the car stopped.
Substituting this expression for t into the equation $-20t^2 + v(0)t = 160$,
$-\frac{[v(0)]^2}{80} + \frac{[v(0)]^2}{40} = 160$, and $v(0) = \pm 80\sqrt{2}$.
The negative sign choice is to be rejected since it requires the stopping time $\frac{v(0)}{40}$ to be negative. Thus $v(0) = 80\sqrt{2}$.
The car was traveling at a rate of $80\sqrt{2}$ ft/s ≈ 113.14 ft/s, or about 124.14 km/m, when the brakes were applied and it stopped in $\frac{v(0)}{40} = 2\sqrt{2}$ s, or about 2.83 s.
This is unrealistic; no one can manage to brake a car on this planet with a deceleration of 40 ft/s^2, about 1.25 times as large as the acceleration due to gravity at the surface of the earth. Even if the brakes were that good, the tires and road surface would not be able to provide a coefficient of friction of 1.25.

Section 9.1 (pg. 531):

4. $x = 2t - 1$, $y = t^2 - 1$.
Solving for t in terms of x, $t = \frac{x+1}{2}$.
$y = \left(\frac{x+1}{2}\right)^2 - 1 = \frac{1}{4}x^2 + \frac{x}{2} - \frac{3}{4}$, a parabola.
See graph to the right.



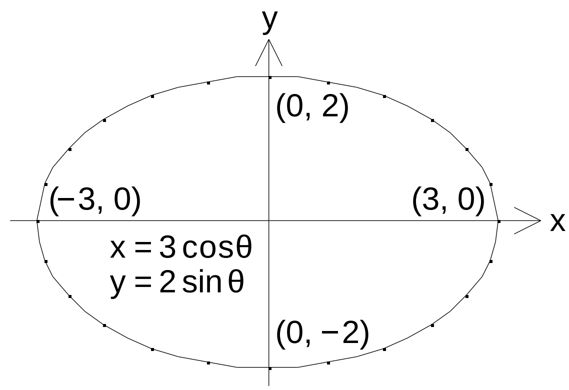
For Exercise 4

8. $x = 3\cos\theta,\ y = 2\sin\theta,\ 0 \le \theta \le 2\pi$.

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2\theta + \sin^2\theta = 1.$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$ an ellipse.

See graph below and to the right.



For Exercise 8

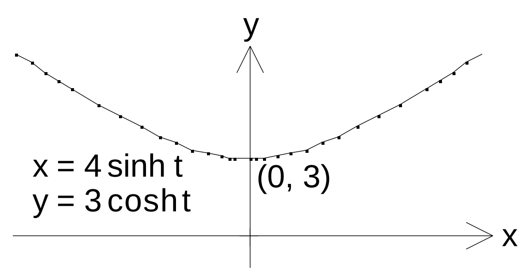
16. $x = 4\sinh t,\ y = 3\cosh t$.

$$\left(\frac{y}{3}\right)^2 - \left(\frac{x}{4}\right)^2 = \cosh^2 t - \sinh^2 t = 1,$$

so $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

We must have $3 \le y < +\infty$,
since that is the range of $y = 3\cosh t$.
So we only get the upper branch of the hyperbola.

See graph below and to the right.



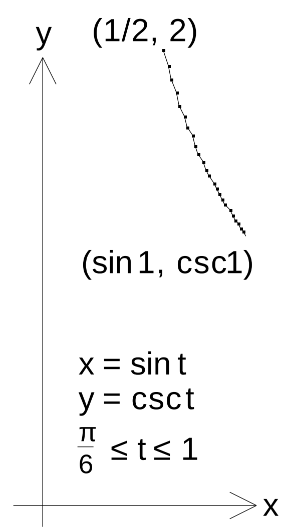
For Exercise 16

22. $x = \sin t,\ y = \csc t,\ \frac{\pi}{6} \le t \le 1$.

$y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$, a hyperbola.

We only obtain the portion of the upper right branch of the hyperbola between $(1/2, 2)$ and $(\sin 1, \csc 1)$.

See graph below and to the right.



For Exercise 22

28. **(a)** $y = (v_0 \sin\alpha)t - \frac{1}{2} g t^2 =$

$= (500 \sin 30°)t - \frac{1}{2}\cdot 9.8 t^2 = 250t - 4.9t^2$.

$y = 0$ when $t = 0$ or $t = \frac{250}{4.9} \approx 51.02040816$.

The bullet will hit the ground in about 51 s.

$x = (v_0 \cos\alpha)t = (500 \cos 30°)t = 250\sqrt{3}\,t$.

When $t = \frac{250}{4.9}$, $x = \frac{62500}{4.9}\sqrt{3} \approx 22092.48479$ m.

The bullet will hit the ground about 22.09 km away.

(b) $y' = 250 - 9.8t$ hence y is maximized when $t = \frac{250}{9.8} \approx 25.51020408$.

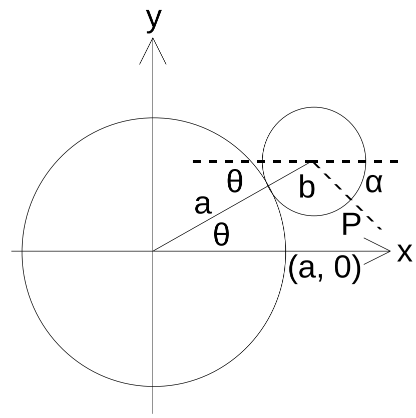
The maximum height above ground realized by the bullet will be about 3.189 km.

(c) $t = \frac{x}{v_0 \cos\alpha}$, while $y = (v_0 \sin\alpha)t - \frac{1}{2} g t^2$.

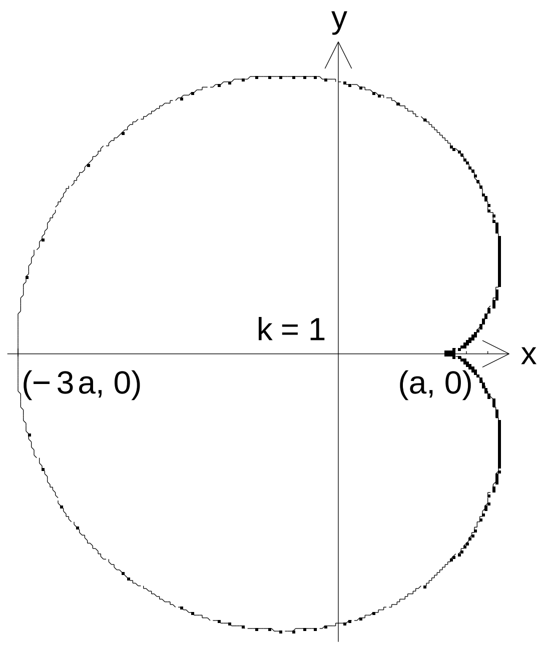
Hence $y = (v_0 \sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = (\tan\alpha)x - \frac{1}{2}\frac{g}{v_0^2\cos^2\alpha} x^2 =$

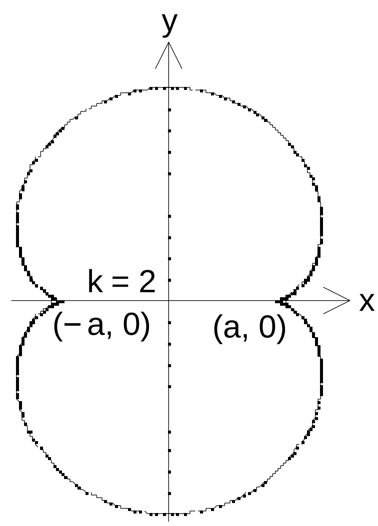
$= \frac{1}{\sqrt{3}} x - \frac{4.9}{62500\ 3} x^2 = \frac{1}{\sqrt{3}} x - \frac{49}{1875000} x^2$. This is an equation of a parabola.

34. (a) Call the initial position of the moving point (a, 0). The small circle has rotated through an arc $a\theta$ but its radius is only b so it has rotated through an angle $\frac{a\theta}{b}$. Thus $\alpha = \left(\frac{a+b}{b}\right)\theta - \pi$.
(Taking α clockwise in the drawing, it is negative.)
The x-coordinate of the point P is
$(a+b)\cos\theta + b\cos\left(\frac{a+b}{b}\,\theta - \pi\right)$, or in other words
$(a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta$, while its y-coordinate is
$(a+b)\sin\theta + b\sin\left(\frac{a+b}{b}\,\theta - \pi\right)$, or in other words
$(a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta$.
See diagram above and to the right.

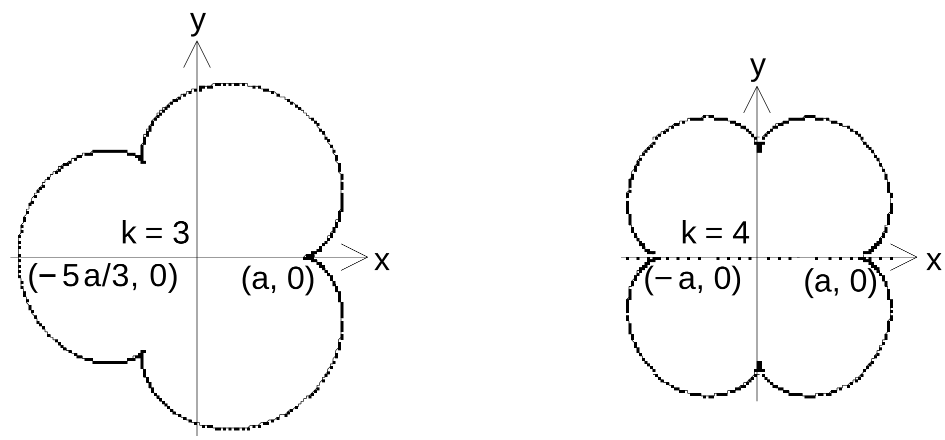


For Exercise 34 (a)

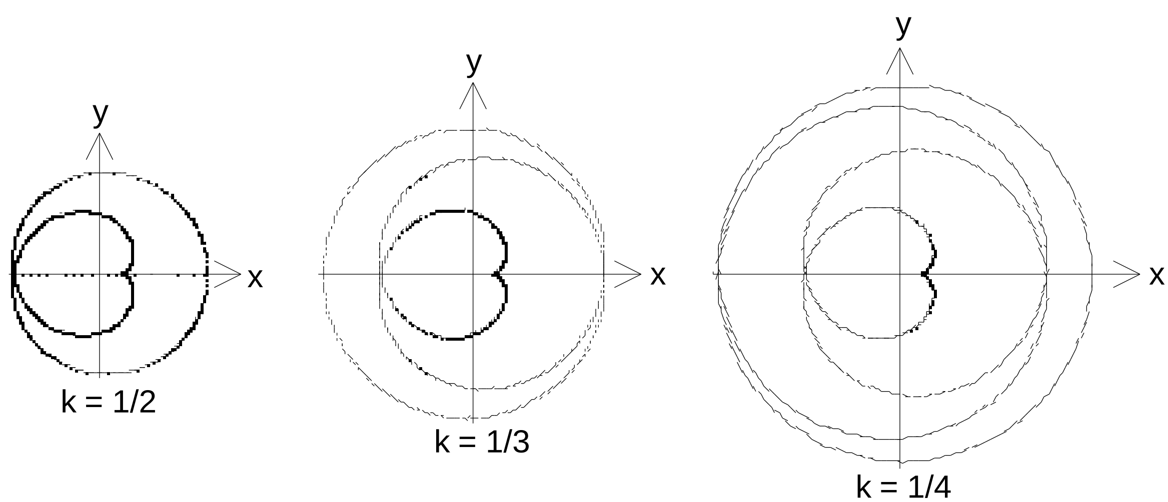




(b) Here are some graphs for various values of $k = \frac{a}{b}$, produced by *Maple*.
The two graphs above are for $k = 1$ and $k = 2$. Graphs for $k = 3$ and $k = 4$ are on the next page. For $k = 1, 2, 3$, or 4 the graph repeats after θ increases by 2π. But for values of k between 0 and 1, the rolling outer circle is of larger radius than the fixed inner circle. This has two important consequence. First, the graphs get larger as k gets smaller. To compensate for this, I have changed the scales by a factor of 5, when $0 < k < 1$. (The graphs for $k \geq 1$ have all been made to the same scale.)
Second, when $k = \frac{1}{2}$ the graph repeats after θ increases by 4π. When $k = \frac{1}{3}$ the graph repeats after θ increases by 6π, and when $k = \frac{1}{4}$ the graph repeats after θ increases by 8π. Below are graphs for $k = 3$, $k = 4$, $k = 1/2$, $k = 1/3$, and $k = 1/4$.



The two graphs above are to the same scale as the ones on the previous page.



These three are to the same scale, 20% as large as the ones for k = 1, 2, 3, and 4.
The cusps on the x-axes just to the right of the origins are all at (a, 0).

Section 9.2 (pg. 538):

4. $x = t\sin t$ and $y = t\cos t$; $\frac{dx}{dt} = \sin t + t\cos t$ and $\frac{dy}{dt} = \cos t - t\sin t$.

At $t = \pi$, $\frac{dx}{dt} = -\pi$ and $\frac{dy}{dt} = -1$, so $\frac{dy}{dx} = \frac{1}{\pi}$.

At the point $(0, -\pi)$ where $t = \pi$, the tangent line has equation $y - (-\pi) = \frac{1}{\pi}(x - 0)$, or $y = \frac{1}{\pi}x - \pi$.

6. $x = 5\cos t$, $y = 5\sin t$. The point $(3, 4)$ corresponds to $t = \tan^{-1}\frac{4}{3} + 2n\pi$, where n can be any integer. For this value of t, $\cos t = \frac{3}{5}$ and $\sin t = \frac{4}{5}$.

(a) $\frac{dx}{dt} = -5\sin t = = -4$ and $\frac{dy}{dt} = 5\cos t = = 3$ at the point $(3, 4)$, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{3}{4}$. The tangent line slope is $-\frac{3}{4}$ and the tangent line has equation $y - 4 = -\frac{3}{4}(x - 3)$.

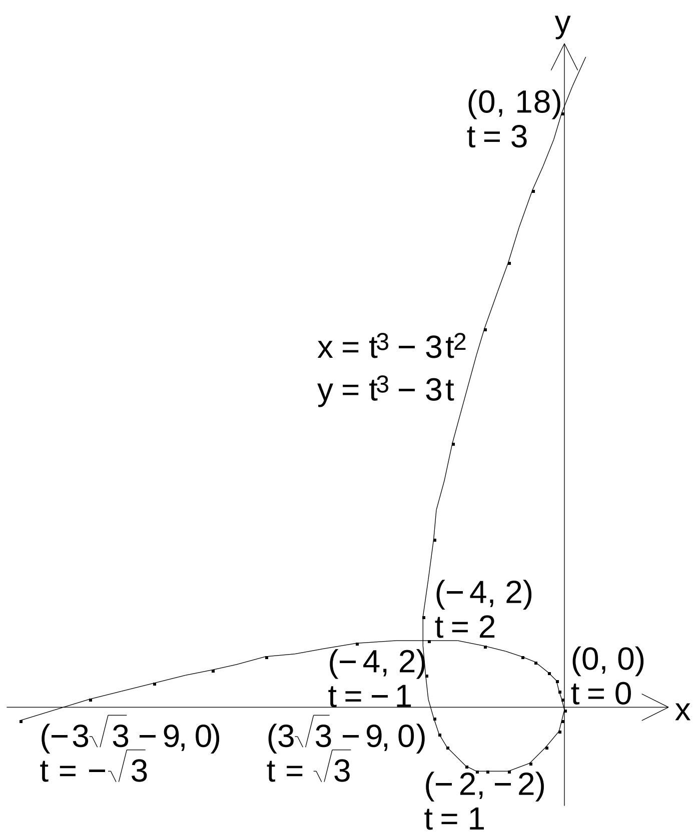
(b) Eliminating the parameter, $x^2 + y^2 = 25$ so $2x + 2y\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{x}{y}$. At the point $(3, 4)$, the tangent line slope is $-\frac{3}{4}$ and the tangent line has equation $y - 4 = -\frac{3}{4}(x - 3)$, as before.

14. $x = 1 + t^2$ and $y = t\ln t$;
$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = \ln t + 1$
so $\frac{dy}{dx} = \frac{\ln t + 1}{2t}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} =$$

$$= \frac{2t\left(\frac{1}{t}\right) - (\ln t + 1)\cdot 2}{8t^3} =$$

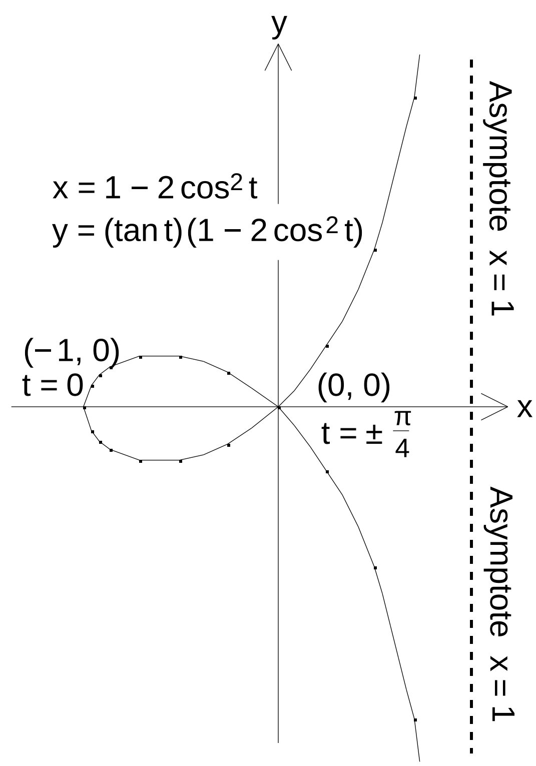
$$= \frac{-2\ln t}{8t^3} = -\frac{\ln t}{4t^3}.$$



For Exercise 16

16. $x = t^3 - 3t^2$ and $y = t^3 - 3t$.
If the tangent is horizontal, $\frac{dy}{dt} = 0$ so $3t^2 - 3 = 0$ and $t = \pm 1$.
If $t = -1$, $(x, y) = (-4, 2)$. If $t = 1$, $(x, y) = (-2, -2)$.
If the tangent is vertical, $\frac{dx}{dt} = 0$ so $3t^2 - 6t = 0$ and $t = 0$ or $t = 2$.
If $t = 0$, $(x, y) = (0, 0)$. If $t = 2$, $(x, y) = (-4, 2)$.
The graph goes to the right and up for $-\infty < t \le -1$, to the right and down for $-1 \le t \le 0$, to the left and down for $0 \le t \le 1$, to the left and up for $1 \le t \le 2$, and to the right and up for $2 \le t < +\infty$.

24. $x = 1 - 2\cos^2 t$; $y = (\tan t)(1 - 2\cos^2 t)$.
The curve meets itself (if $x \neq 0$) for values of t_1 and t_2 where $1 - 2\cos^2 t_1 = 1 - 2\cos^2 t_2 \neq 0$ and $\tan t_1 = \tan t_2$.
Then $t_2 = t_1 + n\pi$ for some integer n.
Conversely if $t_2 = t_1 + n\pi$ then $(\tan t_1)(1 - 2\cos^2 t_1) = (\tan t_2)(1 - 2\cos^2 t_2)$.
The whole curve is periodic with period π.
We won't get different tangents just by retracing the same curve over and over in the same way, so we look for other ways the curve can meet itself.
If $x = 1 - 2\cos^2 t = 0$, then $x = y = 0$, $\cos^2 t = \frac{1}{2}$, and $\cos t = \pm \frac{1}{\sqrt{2}}$, so $t = \pm \frac{\pi}{4} + n\pi$.

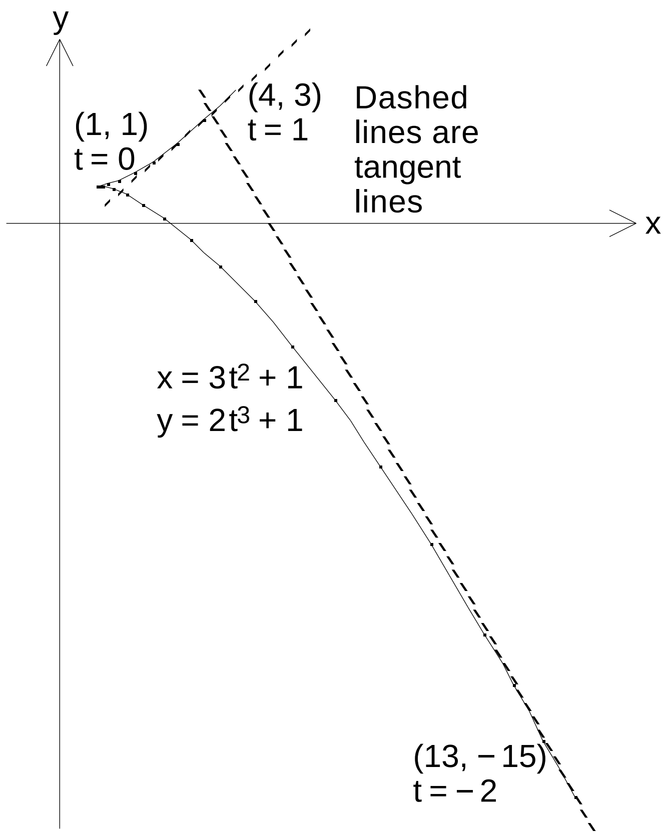


For Exercise 24

$\frac{dy}{dt} = (\sec^2 t)(1 - 2\cos^2 t) + (\tan t)(4\cos t \sin t)$
$= \sec^2 t - 2 + 4\sin^2 t = 2$ and
$\frac{dx}{dt} = 4\cos t \sin t = 2$ if $t = \frac{4n+1}{4}\pi$, so $\frac{dy}{dx} = 1$.
The tangent line is $y = x$ in this case.
$\frac{dy}{dt} = (\sec^2 t)(1 - 2\cos^2 t) + (\tan t)(4\cos t \sin t)$
$= \sec^2 t - 2 + 4\sin^2 t = 2$ but
$\frac{dx}{dt} = 4\cos t \sin t = -2$ if $t = \frac{4n-1}{4}\pi$, so $\frac{dy}{dx} = -1$. The tangent line is then $y = -x$.
See graph above and to the right.

28. If $x = 3t^2 + 1$ and $y = 2t^3 + 1$, $\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2$ so $\frac{dy}{dx} = t$.
For any particular value of t, the tangent to the curve through $(3t^2 + 1, 2t^3 + 1)$ has equation $y - (2t^3 + 1) = t\{x - (3t^2 + 1)\}$, or $y = tx - t^3 - t + 1$.
This passes through $(4, 3)$ when $3 = 4t - t^3 - t + 1$, so $t^3 - 3t + 2 = 0$.
Factoring, $(t + 2)(t - 1)^2 = 0$, and thus either $t = -2$ or $t = 1$.
So the tangent lines in question are $y = -2x + 11$ and $y = x - 1$.
They are tangent to the curve at $(13, -15)$ and at $(4, 3)$ respectively.

See graph on the next page.



For Exercise 28