MATHEMATICS 151

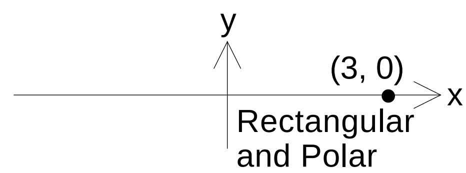
Assignment 14, due Friday 06 August 1999

Section 9.4 (pg. 552):

2. $x = 3 \cos 0 = 3$ and $y = 3 \sin 0 = 0$.

Other polar coordinates: $(3, 2\pi)$, $(-3, \pi)$.
There are infinitely many more:
$(3, 2n\pi)$ and $(-3, (2n+1)\pi)$, n any integer.

See graph to the right.

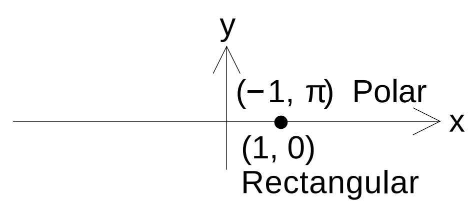


For Exercise 2

6. $x = (-1) \cos \pi = 1$ and $y = (-1) \sin \pi = 0$.

Other polar coordinates: $(1, 0)$, $(-1, 3\pi)$.
There are infinitely many more:
$(1, 2n\pi)$ and $(-1, (2n+1)\pi)$, n any integer.

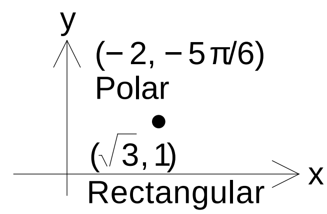
See graph to the right.



For Exercise 6

12. $x = (-2)\cos(-5\pi/6) = \sqrt{3}$
and $y = (-2)\sin(-5\pi/6) = 1$.

See graph below and to the right.

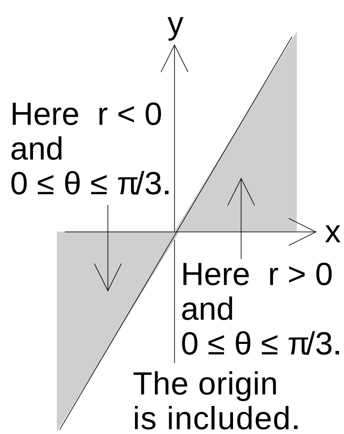


For Exercise 12

16. If $(x, y) = (3, 4)$ and $r > 0$, $r = \sqrt{3^2 + 4^2} = 5$.
If $0 \le \theta < 2\pi$ then $\tan\theta = 4/3$, so $\theta = \tan^{-1}(4/3)$
since $(3, 4)$ is in the first quadrant and $r > 0$. Polar coordinates: $(5, \tan^{-1}(4/3))$.

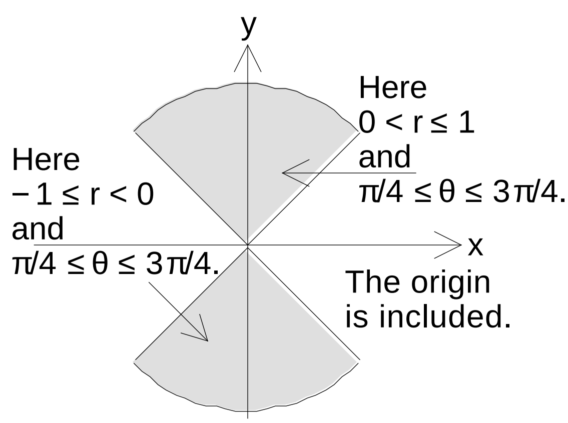
18. $0 \le \theta \le \pi/3$.

See graph below.



22. $-1 \le r \le 1$, $\pi/4 \le \theta \le 3\pi/4$.

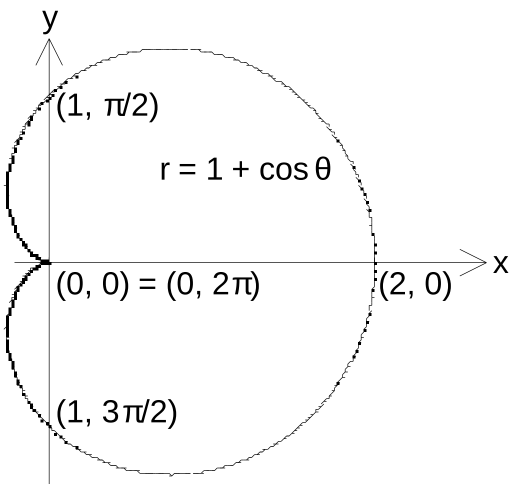
See graph below.



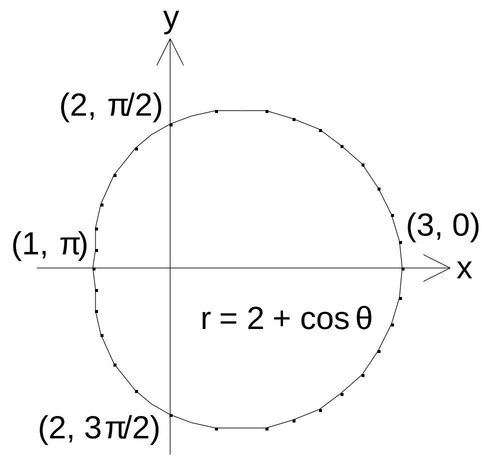
24. $D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1) - r_2 \sin\theta_2)^2} =$

36. $x^2 - y^2 = 1$, so $r^2(\cos^2\theta - \sin^2\theta) = 1$, $r^2\cos(2\theta) = 1$, $r^2 = \sec(2\theta)$, and $r = \pm\sqrt{\sec(2\theta)}$.

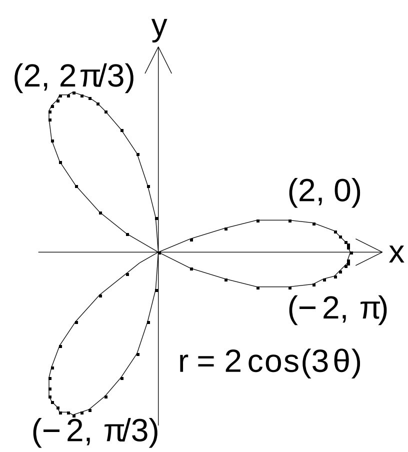
44. $r = 1 + \cos\theta$.
See graph below.



50. $r = 2 + \cos\theta$.
See graph below.



52. $r = 2\cos(3\theta)$. See graph below and to the right.



For Exercise 52

66. $r = \ln\theta$ so $\frac{dr}{d\theta} = \frac{1}{\theta}$, and

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\frac{\sin\theta}{\theta} + \cos\theta\ln\theta}{\frac{\cos\theta}{\theta} - \sin\theta\ln\theta}.$$

At $\theta = e$, $\frac{dy}{dx} = \frac{\frac{\sin e}{e} + \cos e \ln e}{\frac{\cos e}{e} - \sin e \ln e} = \frac{\sin e + e\cos e}{\cos e - e\sin e}$.

70. $r = \cos\theta + \sin\theta$ so $\frac{dr}{d\theta} = -\sin\theta + \cos\theta$.
For a horizontal tangent, $\frac{dr}{d\theta}\sin\theta + r\cos\theta = 0$.
Hence $-\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta + \cos\theta\sin\theta = 0$, or $\cos(2\theta) + \sin(2\theta) = 0$, and $\tan(2\theta) = -1$, so that $\theta = -\frac{\pi}{8} + \frac{n}{2}\pi$, n an integer.
The corresponding points are the points with these values of θ and at which $r = \cos\theta + \sin\theta$.
For a vertical tangent, $\frac{dr}{d\theta}\cos\theta - r\sin\theta = 0$.

Hence $-\cos\theta\sin\theta + \cos^2\theta - \cos\theta\sin\theta - \sin^2\theta = 0$, or $\cos(2\theta) - \sin(2\theta) = 0$, and $\tan(2\theta) = 1$, so that $\theta = \frac{\pi}{8} + \frac{n}{2}\pi$, n an integer.
The corresponding points are the points with these values of θ and at which $r = \cos\theta + \sin\theta$.

It is possible (using the double angle formulas to express $\cos(\pi/4)$ in terms of $\cos(\pi/8)$ or $\sin(\pi/8)$) to find the values of $\cos(\pi/8)$ and $\sin(\pi/8)$ and obtain closed form expressions for the values of r, using radicals.

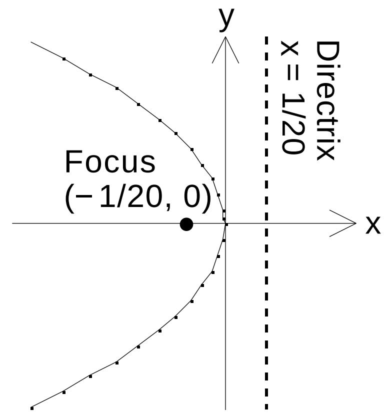
Section 9.6 (pg. 564):

2. $x = -5y^2$ so $y^2 = -\frac{1}{5}x$.

Hence $p = -\frac{1}{20}$.

The vertex is (0, 0), the focus is (− 1/20, 0), and the directrix is x = 1/20.

See graph to the right.



$x = -5y^2$

For Exercise 2

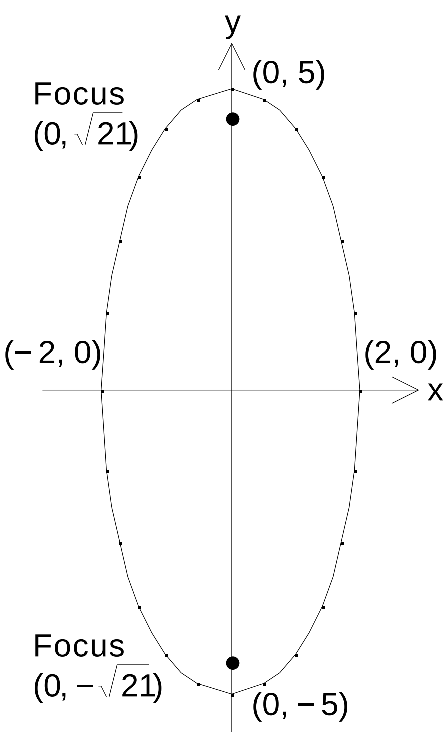
10. $\frac{x^2}{4} + \frac{y^2}{25} = 1$.

The centre is (0, 0).

The vertices are (0, ± 5).

so the foci are $(0, \pm\sqrt{21})$.

See graph below.



$\frac{x^2}{4} + \frac{y^2}{25} = 1$

For Exercise 10

14. $\frac{y^2}{25} - \frac{x^2}{144} = 1$.

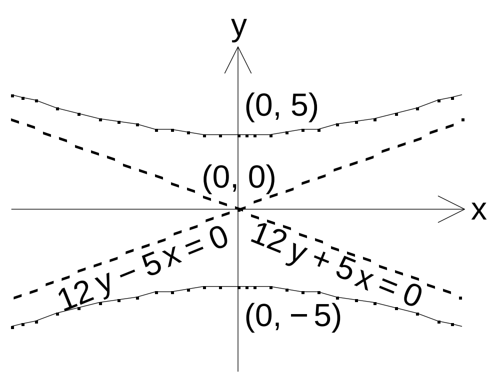
The centre is (0, 0).

The asymptotes are $y = \pm\frac{5}{12}x$.

The vertices are (0, ± 5).

$\sqrt{25 + 144} = 13$ so the foci are (0, ± 13).

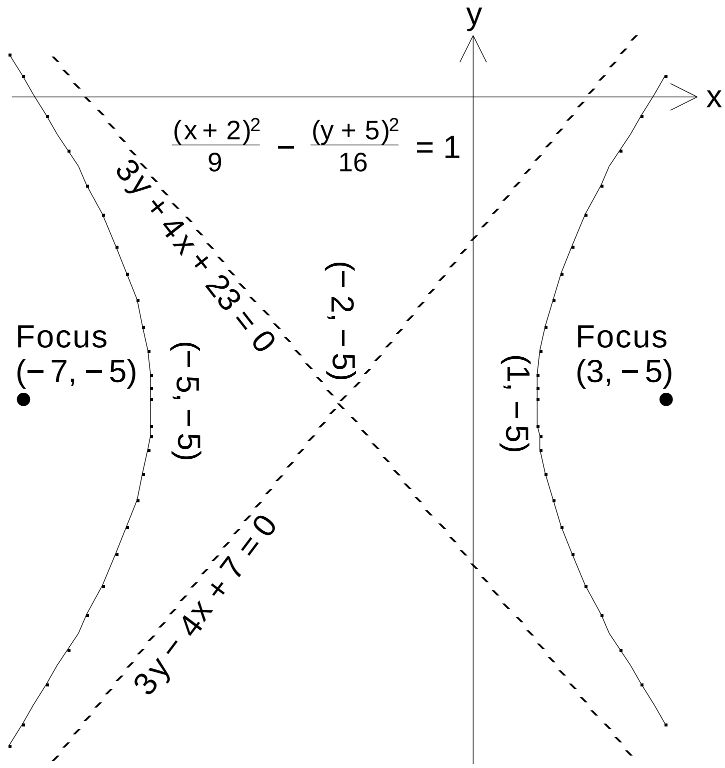
See graph below.



$\frac{y^2}{25} - \frac{x^2}{144} = 1$

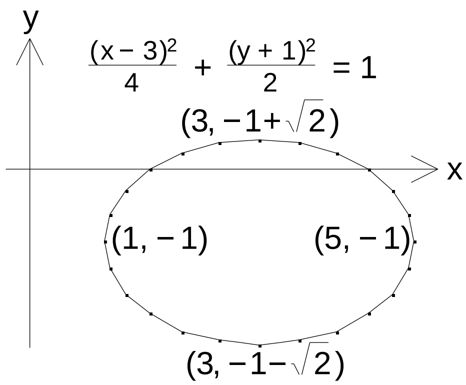
For Exercise 14

18. $16x^2 - 9y^2 + 64x - 90y = 305.$
$16(x+2)^2 - 9(y+5)^2 = 144.$
The centre is $(-2, -5)$.
$\frac{(x+2)^2}{9} - \frac{(y+5)^2}{16} = 1.$
The vertices are $(-5, -5)$ and $(1, -5)$.
The asymptotes are
$y + 5 = \pm \frac{4}{3}(x+2)$, or
$3y - 4x + 7 = 0$ and
$3y + 4x + 23 = 0.$
$\sqrt{9+16} = 5$ so the foci are $(-7, -5)$ and $(3, -5)$.
See graph to the right.

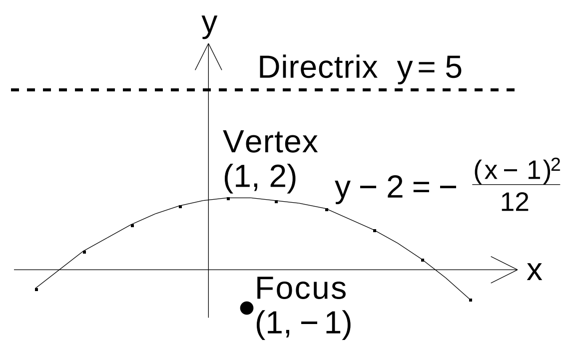


For Exercise 18

20. $x^2 + 2y^2 - 6x + 4y + 7 = 0.$
$(x-3)^2 + 2(y+1)^2 = 4.$
The centre is $(3, -1)$.
$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{2} = 1$, so the vertices are $(1, -1)$ and $(5, -1)$.
$\sqrt{4-2} = \sqrt{2}$ so the foci are $(3-\sqrt{2}, -1)$ and $(3+\sqrt{2}, -1)$.
See graph below.



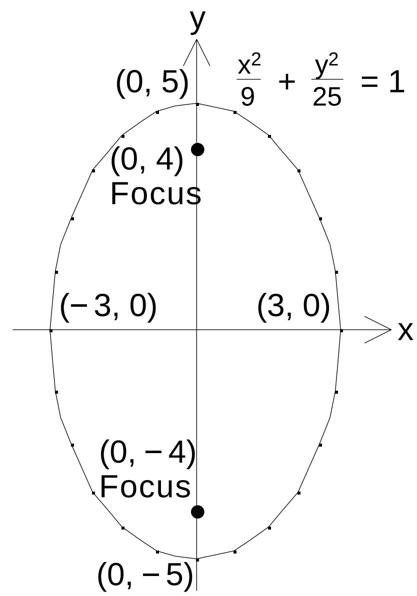
For Exercise 20



For Exercise 24

24. If a parabola has focus $(1, -1)$ and directrix $y = 5$ then its vertex is $(1, 2)$.
Since $p = -3$, its equation is $(x-1)^2 = -12(y-2)$, or $y = -\frac{1}{12}x^2 + \frac{1}{6}x + \frac{23}{12}$.
See graph above.

28. If an ellipse has foci $(0, \pm 4)$ and vertices $(0, \pm 5)$ then its centre is $(0, 0)$. $\sqrt{25-16} = 3$ so the ellipse has equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$.
See graph to the right.

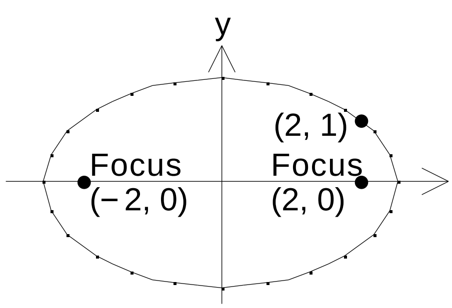


For Exercise 28

32. If an ellipse has foci $(\pm 2, 0)$ its centre is $(0, 0)$.
Suppose its equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
From the focus information, $a^2 - b^2 = 4$,
so the equation is $\frac{x^2}{b^2+4} + \frac{y^2}{b^2} = 1$.
Using the fact that $(2, 1)$ is on the ellipse,
$\frac{4}{b^2+4} + \frac{1}{b^2} = 1$, so $5b^2 + 4 = b^2(b^2+4)$,
$b^4 - b^2 - 4 = 0$, and $b^2 = \frac{1 \pm \sqrt{17}}{2}$.

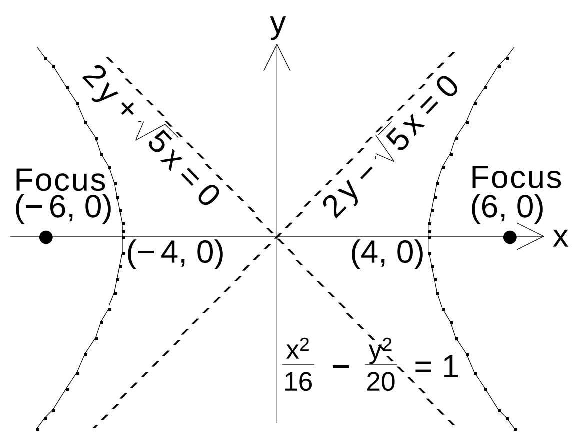
The $-$ sign must be rejected because b^2 cannot be negative, so $b^2 = \frac{1+\sqrt{17}}{2}$ and
$a^2 = b^2 + 4 = \frac{9+\sqrt{17}}{2}$. The ellipse has equation $\frac{x^2}{\frac{9+\sqrt{17}}{2}} + \frac{y^2}{\frac{1+\sqrt{17}}{2}} = 1$.

See graph below.



$$\frac{x^2}{\frac{9+\sqrt{17}}{2}} + \frac{y^2}{\frac{1+\sqrt{17}}{2}} = 1$$

For Exercise 32



For Exercise 34

34. If a hyperbola has foci $(\pm 6, 0)$ and vertices $(\pm 4, 0)$ then its centre is $(0, 0)$.
Since $\sqrt{36-16} = 2\sqrt{5}$, the hyperbola has equation $\frac{x^2}{16} - \frac{y^2}{20} = 1$.
See graph above.

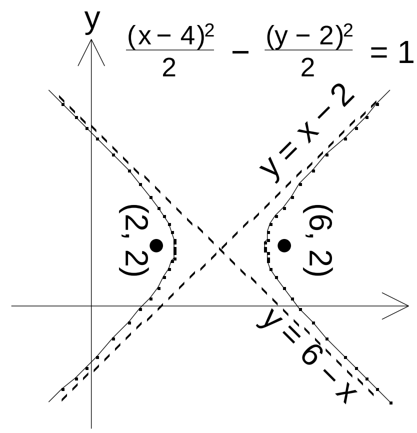
38. If a hyperbola has foci (2, 2) and (6, 2), then its centre must be (4, 2).
The asymptote equations $y = x - 2$ and $y = 6 - x$ may be rewritten as $y - 2 = \pm(x - 4)$.

So the hyperbola's equation is $\frac{(x-4)^2}{a^2} - \frac{(y-2)^2}{a^2} = 1$.

From the focus information, $2a^2 = 4$, so $a = \sqrt{2}$.

The hyperbola has equation $\frac{(x-4)^2}{2} - \frac{(y-2)^2}{2} = 1$.

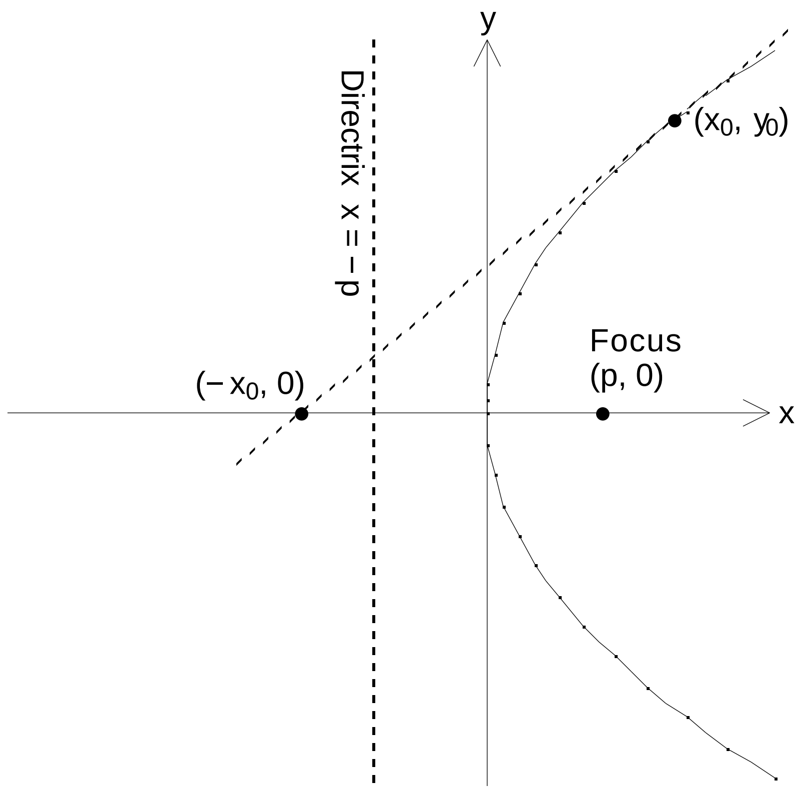
See graph to the right.



For Exercise 38

46. **(a)** If $y^2 = 4px$
then $\frac{dx}{dy} = \frac{2y}{4p} = \frac{y}{2p}$
hence $\frac{dy}{dx} = \frac{2p}{y}$.
At (x_0, y_0), $\frac{dy}{dx} = \frac{2p}{y_0}$
so the tangent line at (x_0, y_0) has equation
$y - y_0 = \frac{2p}{y_0}(x - x_0)$,
or $y_0 y - y_0^2 = 2px - 2px_0$.
But $y_0^2 = 4px_0$,
so the equation is
$y_0 y = 2p(x + x_0)$.

(b) To find the x-intercept of the tangent line, put $y = 0$ in the tangent line equation, obtaining $0 = 2p(x + x_0)$; the x-intercept is $-x_0$.
To obtain the tangent line quickly, draw the line through $(-x_0, 0)$ and (x_0, y_0).



For Exercise 46