Math 314, Boundary Value Problems

J. Hebron, Fall 1999

Final Examination

Monday, Dec 13th, 1999

This is a closed-book, 3-hour exam. No calculators are allowed. Students are allowed one 8.5 by 11 inch formula sheet which may be filled on both sides.

Fill out all the required information on the front of the exam booklet, including your name, student ID number, and signature.

Write all your answers in the exam booklet provided. You may keep the exam questions after you are done. Hand-in only your answer booklet.

The total points for each question are shown in square brackets [like this]. The exam is out of a total of [100] marks, including [3] marks reserved for overall neatness.

Do not open this booklet until told to do so.

1. Consider the following three sets of functions, defined on $-1 < x < 1$:

$$\varphi_n = \sin(n\pi x), \quad n = 1, 2, 3, ...$$
$$\psi_n = \cos(n\pi x), \quad n = 0, 1, 2, ...$$
$$\chi_n = P_n(x), \quad n = 0, 1, 2, ...$$

(a) Each of the sets $\{\varphi\}$, $\{\psi\}$, and $\{\chi\}$ are, within themselves, orthogonal on (-1,1) with a weight function of 1. Write the orthogonality conditions for $\langle\varphi_m|\varphi_n\rangle$, $\langle\psi_m|\psi_n\rangle$, and $\langle P_m|P_n\rangle$, with the appropriate normalization factors. [3]

(b) However, it is not necessarily true that all members of each set are orthogonal to all members of another set. Which subsets are orthogonal? That is, which $\langle\varphi_m|\psi_n\rangle$, $\langle\varphi_m|P_n\rangle$, and $\langle\psi_m|P_n\rangle$ are zero and which aren't? [3]

2. When solving a partial differential equation, under what conditions can one find orthogonal eigenfunctions with a weight function which is not 1? Give a general class of functions, and also give a specific example we've encountered in class. [5]

3. Is there such a thing as a half-order Bessel function? If so, what is it, and where is it used? [3]

4. Is there such a thing as a half-order Legendre Polynomial? If so, what is it and where is it used? [3]

5. Solve the 2-dimensional wave equation in a rectangular membrane of width *a* and height *b*. Choose appropriate boundary conditions. Let the initial velocity be zero. Let the initial displacement be the eigenfunction corresponding to (m,n) = (2,3). Show, neatly and in detail, all the steps of the solution. Finish with a sketch of the nodal lines of the membrane corresponding to your solution. Indicate, with "+" and "–" on your sketch, which areas have an initially positive displacement and which areas have an initially negative displacement. [20]

6. Solve the following potential problem in a cylinder:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < r < a, \qquad 0 < z < b$$
$$u(a,z) = 0, \qquad 0 < z < b$$
$$u(r,0) = 0, \qquad u(r,b) = U_0, \qquad 0 < r < a$$

Show, neatly and in detail, all the steps of the solution. Finish with a rough sketch of the equipotential lines on an *r* versus *z* graph. Make *z* the horizontal axis and make *r* the vertical axis. [20]

7. Consider the following initial value – boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad x > 0, \qquad t > 0$$

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad x > 0$$

$$u(0,t) = h(t), \qquad t > 0$$

where:

$$h(t) = \begin{cases} 1, & 0 < t < 1/c \\ 0, & t > 1/c \end{cases}$$

Find the solution, using D'Alembert's method. Show, neatly and in detail, all the steps of the solution. Finish with accurate sketches of the solution at $t = 1/2c$, $t = 1/c$, and $t = 2/c$. [20]

8. Consider the problem of 1-dimensional heat conduction in a rod in which the cylindrical surface is insulated, the left end is held at a constant temperature T_0, the right end is undergoing convection with a fluid at temperature T_1, and the initial condition is given by $u(x,0) = f(x)$.

- **(a)** Write down the equations governing this system.
- **(b)** What is the steady state solution?
- **(c)** Explain, in words, why the temperature at the right end of the rod isn't T_1 in the steady state solution.
- **(d)** Find the general solution (or as much of it as you can without a computer).

[20]

Some formulas you might need, in case you don't already have them on your formula sheet:

- $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$
- $\int x^{\mu} J_{\mu}(x) x dx = x^{\mu+1} J_{\mu+1}(x)$
- $\frac{d}{dx}\left(x^{-\mu} J_{\mu}(x)\right) = -x^{-\mu} J_{\mu+1}(x)$
- $\frac{d}{dx}\left(x^{\mu} J_{\mu}(x)\right) = x^{\mu} J_{\mu-1}(x)$
- $\int_0^a J_0^2\left(\alpha_n r/a\right) r dr = \frac{a^2}{2} J_1^2(\alpha_n)$