Math 232, Elementary Linear Algebra

J. Hebron, Spring 2000

Final Examination

Tuesday, April 18th, 2000

Time: 3 hours

Student ID Number

Name
(Please underline your family name)

Signature

Instructions:

- **Please fill-in the above information in ink.**
- **Do not open this exam until told to do so.**
- **No books, no calculators, no cell phones.**
- **One 8.5 by 11 inch formula sheet (two-sided) allowed.**
- **The total point value of each problem or part thereof is shown in square brackets.**
- **Please sign the bottom of every page**
  (in case your exam becomes unstapled)

Quest. #:	1	2	3	4	5	6	7	8	9	10	11
Mark:											
Out of:	4	3	2	3	6	2	12	12	6	2	8

Quest. #:	12	13	14	15	16	17	18	19	20	21	Tot
Mark:											
Out of:	12	5	15	5	12	16	7	6	4	8	150

[points]

1. Assume that we live in a 3-dimensional subspace of a 7-dimensional Euclidean space, but we are unaware of the other 4 dimensions, until one day an inventor makes a hyperspace ship capable of travelling in the other dimensions. On the maiden voyage, the hyperspace ship travels from $(0,0,0,0,0,0,0)$ to $(2,-3,4,-2,1,-1,1)$ km.

(a) What is the total distance travelled by the hyperspace ship? [2]

(b) What is the cosine of the angle between the direction travelled by the hyperspace ship and the *x*-axis? (Assume the *x*-axis is in the direction $[1,0,0,0,0,0,0]$.) [2]

2. Let $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots \vec{\mathbf{a}}_5$ be five non-zero non-parallel vectors in $\Re^7$, and let $W = \mathrm{sp}(\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots \vec{\mathbf{a}}_5)$. Suppose W is a 4-dimensional hyperplane in $\Re^7$. Explain how this can be.

[3]

3. Let **A** be a matrix having column vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots \vec{\mathbf{a}}_k \in \Re^n$, and let $\vec{\mathbf{b}} = [b_1, b_2, \ldots b_k] \in \Re^k$. What is $\mathbf{A}\vec{\mathbf{b}}$? [2]

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4. Let **A** be a matrix having column vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots \vec{\mathbf{a}}_k \in \Re^n$, and let **B** be a matrix having row vectors $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots \vec{\mathbf{b}}_m \in \Re^n$. Given that $\vec{\mathbf{a}}_i \bullet \vec{\mathbf{b}}_j = 0$ when $i \neq j$, and $\vec{\mathbf{a}}_i \bullet \vec{\mathbf{b}}_i = 1$, what is **BA**? (Assume $k > m$.) [3]

5. **(a)** What is the elementary matrix which swaps the first and fifth rows of a 7 by 7 matrix?

[2]

(b) What is the elementary matrix which adds 5 times the fifth row of a 5 by 5 matrix to its second row?

[2]

(c) What is the relationship between elementary matrices and invertible matrices?

[2]

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6. Let $W = sp(\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \ldots \vec{\mathbf{w}}_7)$, where $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \ldots \vec{\mathbf{w}}_7 \in \Re^5$, and let **A** be the matrix having column vectors $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \ldots \vec{\mathbf{w}}_7$. Suppose the Row-Echelon form of **A** is:

$$\begin{bmatrix} 5 & 3 & 2 & -4 & -2 & 3 & 5 \\ 0 & -3 & 4 & 1 & -2 & 5 & -1 \\ 0 & 0 & 0 & 1 & -4 & -1 & -4 \\ 0 & 0 & 0 & 0 & -2 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for *W*. [2]

7. Let **B** be a real *n* by *m* matrix and let **C** be a real *m* by *n* matrix, where $m < n$. Let $\mathbf{A} = \mathbf{BC}$.

(a) Prove: $\text{nullity}(\mathbf{A}) \geq n - m$. [10]

(b) If **A** represents a linear transformation **T**, is **T** one-to-one? Why or why not? [2]

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8. The axioms defining an abstract vector space are given as follows:

A1:	$(\vec{\mathbf{u}}+\vec{\mathbf{v}})+\vec{\mathbf{w}}=\vec{\mathbf{u}}+(\vec{\mathbf{v}}+\vec{\mathbf{w}})$	**S1:**	$r(\vec{\mathbf{v}}+\vec{\mathbf{w}})=r\vec{\mathbf{v}}+r\vec{\mathbf{w}}$
A2:	$\vec{\mathbf{v}}+\vec{\mathbf{w}}=\vec{\mathbf{w}}+\vec{\mathbf{v}}$	**S2:**	$(r+s)\vec{\mathbf{v}}=r\vec{\mathbf{v}}+s\vec{\mathbf{v}}$
A3:	$\vec{\mathbf{0}}+\vec{\mathbf{v}}=\vec{\mathbf{v}}$	**S3:**	$r(s\vec{\mathbf{v}})=(rs)\vec{\mathbf{v}}$
A4:	$\vec{\mathbf{v}}+(-\vec{\mathbf{v}})=\vec{\mathbf{0}}$	**S4:**	$1\vec{\mathbf{v}}=\vec{\mathbf{v}}$

The set $\Re^2$ with a weird vector addition "$\oplus$" defined by:

$$[x,y]\oplus[a,b]=[x+a+q,y+b],$$

and a weird scalar multiplication defined by:

$$r[x,y]=[rx+rq-q,ry],$$

where q is a real constant, is a vector space.

(a) Verify axioms **A3**, **A4**, and **S4**.

[10]

(b) Note $r[0,0]=[rq-q,0]\neq[0,0]$. Does this contradict the theorem that $r\vec{\mathbf{0}}=\vec{\mathbf{0}}$? Why or why not? [2]

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9. **(a)** Give an example of a vector space **V** which is not finitely generated. What is its basis?
(*Before answering this question, read parts (b) and (c).*) [2]

(b) Give an example of a proper subspace of **V** which is also not finitely generated. What is its basis?
[2]

(c) Give an example of a proper subspace of **V** which is finitely generated. What is its basis?
[2]

10. Can an n-dimensional vector space be isomorphic to an m-dimensional vector space, where $m \neq n$? Explain.
[2]

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11. Let a polynomial $p(x)$ be defined by the by the following determinant:

$$p(x) = \begin{vmatrix} 1 & a_1 & a_1^2 & a_1^3 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & a_2^3 & \cdots & a_2^n \\ 1 & a_3 & a_3^2 & a_3^3 & \cdots & a_3^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & a_n^3 & \cdots & a_n^n \\ 1 & x & x^2 & x^3 & \cdots & x^n \end{vmatrix}$$

where $a_i \in \Re$ and $a_i \neq a_j$ when $i \neq j$

(a) What is the order of $p(x)$? Explain your answer. [3]

(b) What are the roots of $p(x)$? Explain your answers. [5]

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12. Let $\mathbf{T}: \Re^n \to \Re^n$ be a linear transformation which rotates $\Re^n$ by an angle of $1/2$ radian about the line passing through the origin in the direction $[1, 2, 3, 4, \ldots n]$, and let $\mathbf{A}$ be the standard matrix representation of $\mathbf{T}$.

(a) What are the real eigenvalues and corresponding eigenspaces of $\mathbf{A}$? Explain your answers.

[6]

(b) What are the real eigenvalues and corresponding eigenspaces of $\mathbf{A}^m$, where m is any integer ≥ 2? Explain.

[3]

(c) Is $\mathbf{A}$ an orthogonal matrix? Why or why not? [3]

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13. Find the orthogonal complement to the subspace W of $\Re^3$ defined by $ax + by + cz = 0$, where $a, b, c \in \Re$ are constants and $x, y, z \in \Re$ are variables.

[5]

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14. Let $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_k\}$ be a non-orthogonal set of *k* independent vectors in $\Re^n$, where $k < n$. You are asked to expand this set to form a basis for $\Re^n$. The basis must include $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_k$, so it will not be an orthogonal basis and therefore you can't use the Gram-Schmidt process.

(a) Suppose you want the basis to be composed of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_k$ and $n - k$ standard basis vectors $\hat{\mathbf{e}}_j$. How would you go about finding this basis?

[5]

(b) Let *W* be the *k*-dimensional subspace of $\Re^n$ spanned by $(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_k)$, and let $W^\perp$ be its orthogonal complement. In general, the $n - k$ standard basis vectors found in part (a) will not span $W^\perp$. Suppose you want to find a different basis for $\Re^n$ composed of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots \vec{\mathbf{v}}_k$ and $n - k$ vectors which span $W^\perp$. How would you go about finding this basis?

[5]

(c) Let $\vec{\mathbf{b}} = [b_1, b_2, \ldots b_n]$ be a vector in $\Re^n$. How would you find the projection of $\vec{\mathbf{b}}$ onto *W*?

[5]

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15. Let M_3^u be the vector space of upper triangular real 3 by 3 matrices, and suppose you have an ordered basis given by:

$$B = \left(\begin{bmatrix} a_1 & a_6 & a_5 \\ 0 & a_2 & a_4 \\ 0 & 0 & a_3 \end{bmatrix}, \begin{bmatrix} b_1 & b_6 & b_5 \\ 0 & b_2 & b_4 \\ 0 & 0 & b_3 \end{bmatrix}, \begin{bmatrix} c_1 & c_6 & c_5 \\ 0 & c_2 & c_4 \\ 0 & 0 & c_3 \end{bmatrix}, \begin{bmatrix} d_1 & d_6 & d_5 \\ 0 & d_2 & d_4 \\ 0 & 0 & d_3 \end{bmatrix}, \begin{bmatrix} f_1 & f_6 & f_5 \\ 0 & f_2 & f_4 \\ 0 & 0 & f_3 \end{bmatrix}, \begin{bmatrix} g_1 & g_6 & g_5 \\ 0 & g_2 & g_4 \\ 0 & 0 & g_3 \end{bmatrix} \right)$$

Let $\vec{\mathbf{v}}$ be a vector in M_3^u given by $\vec{\mathbf{v}} = \begin{bmatrix} v_1 & v_6 & v_5 \\ 0 & v_2 & v_4 \\ 0 & 0 & v_3 \end{bmatrix}$.

How would you go about finding $\vec{\mathbf{v}}_B$? [5]

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16. Let $i = \sqrt{-1}$.

(a) What is $\frac{1}{i}$?

[1]

(b) Using Euler's formula, find the three cube roots of -8, in polar form $\left(re^{i\theta}\right)$, and plot them in the complex plane.

[8]

(c) Convert the answers to part (b) into the form $a + ib$, where $a, b \in \Re$.

[3]

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17. Let the matrix **A** be defined by:

$$\mathbf{A} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Find all the eigenvalues of **A**, including the complex ones, and their algebraic multiplicities.

[10]

(b) For all eigenvalues which have an algebraic multiplicity greater than 1, find the corresponding eigenspaces and their geometric multiplicities.

[4]

(c) Is **A** diagonalizable? Explain.

[2]

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18. Let $\mathbf{A}$ be an orthogonal n by n matrix and let $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ be any column vectors in $\Re^n$.

(a) Prove $(\mathbf{A}\vec{\mathbf{x}}) \bullet (\mathbf{A}\vec{\mathbf{y}}) = \vec{\mathbf{x}} \bullet \vec{\mathbf{y}}$.

[3]

(b) Prove $\|\mathbf{A}\vec{\mathbf{x}}\| = \|\vec{\mathbf{x}}\|$.

[2]

(c) Prove the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ is the same as the angle between $\mathbf{A}\vec{\mathbf{x}}$ and $\mathbf{A}\vec{\mathbf{y}}$.

[2]

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19. Suppose you have a computational algorithm which processes large amounts of measured data. Let the data be placed into a vector $\vec{\mathbf{x}}$ in $\Re^n$ and let **A** be an n by n matrix which multiplies $\vec{\mathbf{x}}$ in the computation. What is the advantage of orthogonalizing the computational algorithm so that **A** is an orthogonal matrix?

[6]

20. The projection matrix **P** for the subspace W of $\Re^n$ spanned by the column vectors of **A** is given by $\mathbf{P} = \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$. What's wrong with the following argument?
$\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T = \mathbf{A}\left(\mathbf{A}^{-1}\left(\mathbf{A}^T\right)^{-1}\right)\mathbf{A}^T = \left(\mathbf{A}\mathbf{A}^{-1}\right)\left(\left(\mathbf{A}^T\right)^{-1}\mathbf{A}^T\right) = (\mathbf{I})(\mathbf{I}) = \mathbf{I}$, and therefore $\mathbf{P} = \mathbf{I}$.
Explain your reasoning.

[4]

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21. Let $V = \text{sp}\left(e^{-x}, 1, e^{x}\right)$ be a subspace of the vector space of continuous functions of one variable, and let $B = \left(e^{-x}, 1, e^{x}\right)$ be an ordered basis for V. Let $B' = (\cosh(x), \sinh(x), 1)$ be another ordered basis for V. What is the change of coordinates matrix $C_{B,B'}$?

[8]

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