SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

Final Exam

MATH 232

August 6, 1998, 8:30 – 11:30 p.m.

Name: ______________________________
family name *given name*

Number: ______________________________

INSTRUCTIONS

1. This exam has 14 questions on 19 pages. Please check to make sure your exam is complete.
2. No calculators may be used.
3. Ask for clarification if you cannot understand the question or there appears to be an error.
4. Please write with a black or blue pen.
5. Use the reverse sides of the pages for rough work.
6. In each question indicate how you obtain your answer.

   The more clearly you explain what you are doing the more part marks you will get.

Question	Score	Max
1		7
2		7
3		4
4		6
5		6
6		8
7		7
8		7
9		8
10		7
11		8
12		8
13		8
14		9
Total		100

[4] **1.** (a) **Describe carefully the kinds of row operation which are permitted in bringing a matrix to reduced row-echelon form.**

ANSWER BOX

[3] (b) **Find a reduced row-echelon matrix row-equivalent to**

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \end{bmatrix}$$

ANSWER BOX

2. Let

$$A = \begin{bmatrix} 0 & 0 & -3 & -4 & -5 \\ 1 & -1 & -2 & -3 & -4 \\ 0 & 0 & -1 & -2 & -3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

[4] (a) **Express the general solution of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ as a linear combination of vectors in $\mathbb{R}^5$.**

ANSWER BOX

[3] (b) **Find a particular solution of the nonhomogeneous system $A\boldsymbol{x} = \boldsymbol{b}$.**

The solution should be in terms of b_1, b_2, b_3, but contain no arbitrary constants.

ANSWER BOX

[4] **3.** Let A denote the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Express A as a product of elementary matrices.

ANSWER BOX

4. Let $A \in \mathbb{R}^{5\times 6}$ have columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_6$ respectively. Let the reduced row-echelon form of A be

$$H = \begin{bmatrix} 1 & 0 & -2 & 0 & -3 & -4 \\ 0 & 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[2] (a) **Write down a basis for the row space of A.**

ANSWER BOX

[2] (b) **Write down a basis for the column space of A.**

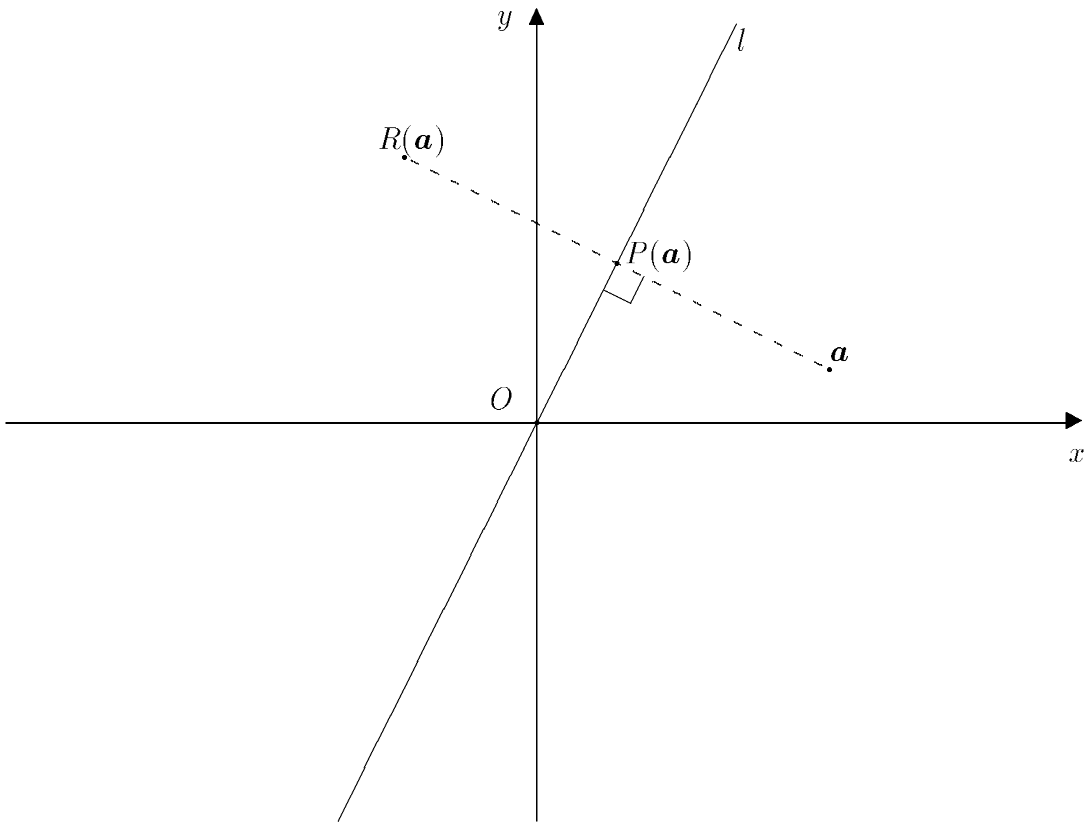
ANSWER BOX

[2] (c) **Write down a basis for the nullspace of A.**

ANSWER BOX

5. Let l be the line in $\mathbb{R}^2$ whose equation is $y = 2x$. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $P(\boldsymbol{a})$ is the projection of $\boldsymbol{a}$ on l.

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $R(\boldsymbol{a})$ is the reflection of $\boldsymbol{a}$ in l.



[3] (a) **Show that P is a linear transformation and find the matrix which is its standard representation.**

Write your answer in the box provided on the next page.

[3] (b) **Express R as a linear combination of P and I the identity transformation on $\mathbb{R}^2$.**

Hence show that R is also a linear transformation.

Write your answer in the box provided on the next page.

ANSWER BOX

(a)

ANSWER BOX

(b)

[4] **6.** (a) Let V be a vector space over $\mathbb{R}$ and S be a subset of V.

State a criterion for S to be a subspace of V.

ANSWER BOX

[4] (b) **Which of the following sets are subspaces of $\mathbb{R}^3$?**

1. $S = \{[x, y, z] : x \geq 0\}$
2. $S = \{[x, y, z] : x + 3y = z\}$
3. $S = \{[x, y, z] : xy = 0\}$

Justify your answers briefly.

ANSWER BOX

7. Let $V = \mathbb{R}^{2\times 2}$ denote the vector space over $\mathbb{R}$ whose vectors are the 2×2 matrices with entries from $\mathbb{R}$. Let

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ \boldsymbol{v}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

[4] (a) **Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly independent in V.**

ANSWER BOX

[3] (b) **Find the coordinate vector of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with respect to the ordered basis $\mathcal{B} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle$.**

ANSWER BOX

[4] **8.** (a) **Evaluate the determinant**

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix}$$

ANSWER BOX

[3] (b) Let A be an $n \times n$ matrix and A_{ij} denote the i, j-th minor of A.

State the formula which expands $\det(A)$ **by the** j**-th column.**

ANSWER BOX

9. Let A denote the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

[4] (a) **Find the eigenvalues of A.**

ANSWER BOX

[4] (b) **Discover whether A is diagonalizable over and $\mathbb{R}$ explain your answer carefully.**

ANSWER BOX

[3] **10.** (a) **Find a basis for the orthogonal complement in $\mathbb{R}^4$ of the space** $W = \mathbf{sp}([1, 2, -1, 1], [1, -1, 1, 1])$.

ANSWER BOX

[4] (b) Let $\boldsymbol{b} = [0, 3, 3, 3]$.

Find the projection $\boldsymbol{b}_W$ of $\boldsymbol{b}$ on W.

ANSWER BOX

11. Let $W = \text{sp}([1,-1,-1,1],[2,2,1,1],[0,1,0,1])$.

[5] (a) **Find an orthogonal basis for W.**

ANSWER BOX

[3] (b) **Find an orthonormal basis for W.**

ANSWER BOX

12. Let $\boldsymbol{b}$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be nonzero vectors in $\mathbb{R}^n$ such that

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0 \qquad (1 \leq i < j \leq k).$$

Let W denote $\text{sp}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$ and $\boldsymbol{b}_W$ denote the projection of $\boldsymbol{b}$ on W.

Let A denote the $n \times k$ matrix $[\boldsymbol{v}_1 \; \ldots \; \boldsymbol{v}_k]$ whose columns are $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

[4] (a) **Prove that the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is linearly independent.**

ANSWER BOX

[4] (b) **Prove that**

$$\boldsymbol{b}_W = A\left(A^T A\right)^{-1} A^T \boldsymbol{b}\,.$$

What has to be shown here is that the vector $\boldsymbol{b}_W$ defined by this formula satisfies $\boldsymbol{b}_W \in W$ and $\boldsymbol{b} - \boldsymbol{b}_W \perp W$.

ANSWER BOX

13. Let V denote the vector space over $\mathbb{R}$ consisting of all polynomials in $\mathbb{R}[x]$ of degree at most 2.
Let

$$\mathcal{B} = \langle x+1, (x+1)^2, 1\rangle, \quad \mathcal{B}' = \langle 2x-1, 2x+1, x^2+x\rangle.$$

Let $F : V \to V$ be the unique linear transformation such that $F(1) = 1 + x$, $F(x) = x + x^2$, $F(x^2) = 1 + x^2$.

[4] (a) **Find a matrix $C \in \mathbb{R}^{3\times 3}$ such that, for all $\boldsymbol{v}$ in V,**

$$C\boldsymbol{v}_{\mathcal{B}} = \boldsymbol{v}_{\mathcal{B}'}\,.$$

ANSWER BOX

[4] (b) **Find the matrix $[F]_{\mathcal{B}',\mathcal{B}'}$ which represents F with respect to $\mathcal{B}'$, $\mathcal{B}'$.**

ANSWER BOX

[9] **14.** Consider the surface S in $\mathbb{R}^3$ whose equation is

$$-x^2 + y^2 - z^2 - 2yz - 6zx + 2xy = 1\,.$$

Determine as well as you can the nature of the surface S.

Some questions that should be addressed are:

Does S have a centre? Is S connected? Are there axes?

Make it clear how you reach your conclusions.

ANSWER BOX