SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

Sample Final Exam

MATH 232

August 1998, 3 hours

Name: ______________________________

family name *given name*

Number: ______________________________

INSTRUCTIONS

1. This exam has 15 questions on 17 pages. Please check to make sure your exam is complete.
2. No calculators may be used.
3. Please write with a black or blue pen.
4. Use the reverse sides of the pages for rough work.
5. In each question indicate how you obtain your answer. Unless the question specifically requires it, you need not give details.

Question	Score	Max
1		7
2		7
3		6
4		7
5		6
6		6
7		6
8		7
9		8
10		7
11		6
12		8
13		8
14		5
15		6
Total		100

[4] **1.** (a) **Define the term "reduced row-echelon matrix".**

ANSWER BOX

[3] (b) **Find a reduced row-echelon matrix row-equivalent to**

$$\begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 2 \\ 4 & -4 & 4 \end{bmatrix}$$

ANSWER BOX

2. Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \\ \boldsymbol{x}_4 \\ \boldsymbol{x}_5 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

[4] (a) Express the general solution of the homogeneous system $\boldsymbol{Ax} = \mathbf{0}$ as a linear combination of vectors in $\mathbb{R}^5$.

ANSWER BOX

[3] (b) Write down the general solution of the nonhomogeneous system $\boldsymbol{Ax} = \boldsymbol{b}$.

ANSWER BOX

3. The matrix $\boldsymbol{A}$ in $\mathbb{R}^{n\times n}$ is defined to be *invertible* if there exists $\boldsymbol{B}$ in $\mathbb{R}^{n\times n}$ such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}\,.$$

Let $\boldsymbol{A}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$ both be invertible.

[3] (a) **Show that $\boldsymbol{AC}$ is invertible.**

ANSWER BOX

[3] (b) **Show that $\boldsymbol{A}^T$ is invertible.**

ANSWER BOX

[3] **4.** (a) **Explain how to compute the rank of a matrix.**

ANSWER BOX

[4] (b) **Explain why it is true that**

$$\operatorname{rank}(\boldsymbol{AB}) \leq \min(\operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{B}))$$

for all matrices $\boldsymbol{A}$, $\boldsymbol{B}$ such that $\boldsymbol{AB}$ is defined.

ANSWER BOX

[6] **5. Find a formula which defines a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies**

$$F([1,1,1]) = [0,1,1], \qquad F([1,-1,-1]) = [1,0,0]\,.$$

ANSWER BOX

Definition of vector space

A *vector space over* $\mathbb{R}$ is a set $\boldsymbol{V}$ of vectors together with a distinguished vector $\mathbf{0}$ in $\boldsymbol{V}$ and three functions

$$(\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} + \boldsymbol{v} \qquad (\boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{V})$$

$$\boldsymbol{v} \mapsto -\boldsymbol{v} \qquad (\boldsymbol{v} \in \boldsymbol{V})$$

$$(\boldsymbol{r}, \boldsymbol{v}) \mapsto \boldsymbol{r}\boldsymbol{v} \qquad (\boldsymbol{r} \in \mathbb{R},\ \boldsymbol{v} \in \boldsymbol{V})$$

which satisfy

A1	$(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$	associative law
A2	$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$	commutative law
A3	$\mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$	additive identity
A4	$\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$	additive inverse
S1	$\boldsymbol{r}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{r}\boldsymbol{u} + \boldsymbol{r}\boldsymbol{v}$	distributivity
S2	$(\boldsymbol{r} + \boldsymbol{s})\boldsymbol{u} = \boldsymbol{r}\boldsymbol{u} + \boldsymbol{s}\boldsymbol{u}$	distributivity
S3	$\boldsymbol{r}(\boldsymbol{s}\boldsymbol{u}) = (\boldsymbol{r}\boldsymbol{s})\boldsymbol{u}$	associative law
S4	$\mathbf{1}\boldsymbol{u} = \boldsymbol{u}$	scale preservation

for all $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in \boldsymbol{V}$ and $\boldsymbol{r},\ \boldsymbol{s} \in \mathbb{R}$.

[6] **6. From the axioms for a vector space show that for all vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in $\boldsymbol{V}$,**

$$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{w} \;\Rightarrow\; \boldsymbol{v}=\boldsymbol{w}\,.$$

ANSWER BOX

7. Let $\boldsymbol{V} = \mathbb{R}^{2\times 2}$ denote the vector space over $\mathbb{R}$ whose vectors are the $\mathbf{2} \times \mathbf{2}$ matrices with entries from $\mathbb{R}$. Let

$$\boldsymbol{v}_1 = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \; \boldsymbol{v}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}, \; \boldsymbol{v}_3 = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \; \boldsymbol{v}_4 = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

[3] (a) **Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly dependent.**

ANSWER BOX

[3] (b) **Find $\boldsymbol{u} \in \boldsymbol{V}$ such that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{u}\}$ is a basis for $\boldsymbol{V}$.**

ANSWER BOX

[4] **8.** (a) **Evaluate the determinant**

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} & \mathbf{0} & \mathbf{5} & \mathbf{0} \\ \mathbf{6} & \mathbf{0} & \mathbf{7} & \mathbf{0} & \mathbf{8} \\ \mathbf{0} & \mathbf{9} & \mathbf{0} & \mathbf{10} & \mathbf{0} \\ \mathbf{11} & \mathbf{0} & \mathbf{12} & \mathbf{0} & \mathbf{13} \end{bmatrix}$$

ANSWER BOX

[3] (b) Let $\boldsymbol{A}$ be a square matrix.

State the relationship between det($\boldsymbol{A}$) and rank($\boldsymbol{A}$).

ANSWER BOX

9. Let $\boldsymbol{A}$ denote the matrix

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

[4] (a) **Find the eigenvalues of $\boldsymbol{A}$.**

ANSWER BOX

[4] (b) **Find a matrix C such that $C^{-1}AC$ is a diagonal.**

ANSWER BOX

[3] **10.** (a) **Find the projection of $[2, -1, 3]$ on $\text{sp}([1, 2, -1])$.**

ANSWER BOX

[4] (b) **Find a formula for the projection of $b = [b_1, b_2, b_3]$ on the subspace $\text{sp}([1, 1, -1], [-1, 1, 1])$.**

ANSWER BOX

[3] **11.** (a) **State three conditions on a matrix $A \in \mathbb{R}^{n\times n}$ which are equivalent to A being an orthogonal matrix.**

ANSWER BOX

[3] (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal linear transformation such that

$$T([1,0]) = [1/2, \sqrt{3}/2].$$

Explain why there are only two possibilities for T and describe them.

ANSWER BOX

[8] **12.** The following data points are given:

$$(-2,-8),\ (-1,-8),\ (2,0),\ (3,0),\ (4,2),\ (6,8)$$

By using a method from linear algebra find the least-squares linear fit for these data points.

Your answer should make it clear what method you are using.

ANSWER BOX

13. Let ${}^{\mathbb{R}}\mathbb{R}$ denote the vector space over $\mathbb{R}$ consisting of all functions $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}$. Let $\boldsymbol{V}$ denote the subspace of ${}^{\mathbb{R}}\mathbb{R}$. spanned by $\{\mathbf{1}, \sin 2x, \cos 2x\}$. Let

$$\mathcal{B} = \langle 1, \sin 2x, \cos 2x \rangle, \quad \mathcal{B}' = \langle \sin^2 x, \cos^2 x, \sin x \cos x \rangle.$$

Let $F : V \to V$ be the unique linear transformation which maps $\mathcal{B}$ to $\mathcal{B}'$ in the sense that $F(1) = \sin^2 x$, $F(\sin 2x) = \cos^2 x$, and $F(\cos 2x) = \sin x \cos x$.

[4] (a) **Find a matrix $C \in \mathbb{R}^{3\times 3}$ such that, for all v in V,**

$$C v_{\mathcal{B}} = v_{\mathcal{B}'}$$

ANSWER BOX

[4] (b) **Find the matrix $[F]_{\mathcal{B},\mathcal{B}}$ which represents F with respect to $\mathcal{B}$, $\mathcal{B}$.**

ANSWER BOX

[5] **14.** Consider the curve in $\mathbb{R}^2$ whose equation is

$$6x^2 + \sqrt{24}xy + 7y^2 = 1\ .$$

Show that this curve is an ellipse and find the length of its major and minor axes.

ANSWER BOX

[6] **15.** Consider the surface S in $\mathbb{R}^3$ whose equation is

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 3x + z = 1\,.$$

Show that S is cylindrical in the sense that there is a unit vector u such that S is invariant under translation by any scalar multiple of u.

ANSWER BOX

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