SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

Final Exam

MATH 232

April 8, 1999, 8:30-11:30 a.m.

Name: ______________________________

family name *given name*

Number: ______________________________

Question	Score	Max
1		7
2		7
3		8
4		8
5		6
6		4
7		6
8		4
9		10
10		8
11		10
12		10
13		6
14		6
Total		100

INSTRUCTIONS

1. This exam has 14 questions on 12 pages. Please check to make sure your exam is complete.
2. Write your final answer in the answer box when one is provided.
3. No calculators or other computing devices may be used.
4. Please write with a black or blue pen.
5. If you need more room, use the reverse side of the **previous page** to show your work.
6. In each question indicate how you obtain your answer. You may lose points if your work is poorly presented.

[4] **1.** (a) **Define the term "reduced row-echelon matrix".**

ANSWER BOX

[3] (b) **Find a reduced row-echelon matrix row-equivalent to**

$$\begin{bmatrix} 0 & 2 & 1 & -1 \\ 1 & 1 & -2 & 1 \\ -1 & 1 & 3 & -1 \end{bmatrix}$$

ANSWER

SHOW YOUR WORK

[7] **2.** Let

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & -1 & -1 \\ 0 & 2 & 1 & 2 \\ 1 & 4 & 1 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = [1, -2, 1, 0]$$

Find the general solution of the system $A\boldsymbol{x} = \boldsymbol{b}$.

ANSWER

SHOW YOUR WORK

3. Let V denote the subspace of $\mathbb{R}^6$ spanned by

$$\begin{aligned} \boldsymbol{a}_1 &= [2, -1, 3, 4, 1, 2] \\ \boldsymbol{a}_2 &= [-2, 5, 3, 2, 1, -4] \\ \boldsymbol{a}_3 &= [2, 4, 6, 5, 2, 1] \\ \boldsymbol{a}_4 &= [1, -1, 1, -1, 2, 2] \\ \boldsymbol{a}_5 &= [1, 8, 10, 2, 5, -1] \\ \boldsymbol{a}_6 &= [3, 0, 0, 2, 1, 5] \end{aligned}$$

and $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 \end{bmatrix}$ be the 6×6 matrix whose columns are $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6$.

By elementary row operations A is converted to

$$H = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[2] (a) **What is the dimension of V?**

ANSWER

[3] (b) **Write down a basis for V.**

ANSWER

[3] (c) **Write down a basis for the nullspace of A.**

ANSWER

ROUGH WORK IF REQUIRED

(use the back of page 2)

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation.

[2] (a) **Define ker**(T).

ANSWER

[2] (b) **Define range**(T).

ANSWER

[4] (c) It is given that

$$T([1,0,0,0]) = [1,2,3], \qquad T([1,1,0,0]) = [2,3,4]$$
$$T([1,1,1,0]) = [3,4,5], \qquad T([1,1,1,1]) = [4,5,6]\,.$$

Find the standard matrix representation of T.

ANSWER

SHOW YOUR WORK

[6] **5.** On a separate sheet circulated with the exam you have the definition of a *vector space over* $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$.

From the axioms listed on the sheet, prove that, for all vectors a, in V,

$$-\boldsymbol{a} = (-1)\boldsymbol{a}\,.$$

Hint: one might start by proving that $0\boldsymbol{a} = \mathbf{0}$ for all $\boldsymbol{a} \in V$.

ANSWER

[4] **6.** Let ${}^{\mathbb{R}}\mathbb{R}$ denote the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ and

$$V = \left\{ f \in {}^{\mathbb{R}}\mathbb{R} : (\forall x,\, y \in \mathbb{R})\,(xy > 0 \text{ implies } f(x) = f(y)) \right\}\,.$$

Find a basis of V as a subspace of ${}^{\mathbb{R}}\mathbb{R}$.

ANSWER

[6] **7.** Let $A \in \mathbb{R}^{n\times n}$ and ρ be an elementary row operation.

Describe how $\det \rho(A)$ depends on ρ and $\det(A)$.

ANSWER

[4] **8. Evaluate the determinant**

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & 1 & 2 \\ 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 & 3 \end{bmatrix}$$

ANSWER

SHOW YOUR WORK

9. Let A denote the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

[4] (a) **Find the eigenvalues of A.**

ANSWER

[4] (b) **Find the eigenspaces of A.**

ANSWER

[2] (c) **Is A diagonalizable?**

Justify your answer.

ANSWER

SHOW YOUR WORK

10. Let A denote the matrix $\begin{bmatrix} 3/10 & 4/10 & 3/10 \\ 1/10 & 2/10 & 1/10 \\ 6/10 & 4/10 & 6/10 \end{bmatrix}$.

It is given that the eigenvalues of A are $0,\ 1,\ 1/10$.

[4] (a) **Find C such that $C^{-1}AC$ is a diagonal matrix.**

ANSWER

[4] (b) **Compute $\lim_{n\to\infty} A^n$.**

ANSWER

SHOW YOUR WORK

11. Let W denote the subspace of $\mathbb{R}^4$ defined by $W = \text{sp}([1,1,-1,1],[1,1,0,0])$.
Let $\boldsymbol{b} = [b_1, b_2, b_3, b_4]$ be a general vector in $\mathbb{R}^4$.

[3] (a) **Find the orthogonal complement $W^\perp$ of W.**

ANSWER

[3] (b) **Find an orthogonal basis for W.**

ANSWER

[4] (c) **Find $\boldsymbol{b}_W$ the projection of $\boldsymbol{b}$ on W. Your answer should give the components of $\boldsymbol{b}_W$ explicitly in terms of b_1, b_2, b_3, b_4.**

ANSWER

SHOW YOUR WORK

12. Let

$$B_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \quad B_3 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

and

$$\mathcal{B} = \langle B_1,\, B_2,\, B_3 \rangle \qquad \mathcal{B}' = \langle B_4,\, B_5,\, B_6 \rangle\,.$$

Let V denote the subspace of $\mathbb{R}^{2\times 2}$ of which $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases.
Let $T : V \to V$ be the linear operator defined by

$$T(X) = X \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (X \in V)\,.$$

[5] (a) **Compute the change of basis matrix $C_{\mathcal{B},\mathcal{B}'}$.**

ANSWER

[5] (b) **Compute the matrix $[T]_{\mathcal{B}}$ which represents T with respect to $\mathcal{B}$.**

ANSWER

SHOW YOUR WORK

[6] **13.** **Find a rotation of $\mathbb{R}^3$ which diagonalizes the quadratic form**

$$2x^2 + 3y^2 + 2z^2 + 2xy + 2yz.$$

ANSWER

SHOW YOUR WORK

[6] **14.** **Explain briefly the role of diagonalization of** 2×2 **matrices in classifying curves in** $\mathbb{R}^2$ **whose equations have the form**

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

with $a,\ b,\ c,\ d,\ e,\ f \in \mathbb{R}$.

WRITE YOUR ANSWER HERE