MATH 232 Key for Final Exam, August 6, 1998

[4] **1.** (a) **Describe carefully the kinds of row operation which are permitted in bringing a matrix to reduced row-echelon form.**

ANSWER BOX

Let the matrix under consideration have m rows and entries from $\mathbb{R}$. The permissible row operations are

1. switch rows i and j
2. replace row i by row i plus c(row j), where $c \in \mathbb{R} - \{0\}$
3. replace row i by c(row i), where $c \in \mathbb{R} - \{0\}$.

■

[3] (b) **Find a reduced row-echelon matrix row-equivalent to**

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \end{bmatrix}$$

ANSWER BOX

Applying the indicated row operations we have

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - R_1,\ R_3 \to R_3 - 2R_2$$

■

2. Let

$$A = \begin{bmatrix} 0 & 0 & -3 & -4 & -5 \\ 1 & -1 & -2 & -3 & -4 \\ 0 & 0 & -1 & -2 & -3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

[4] (a) **Express the general solution of the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ as a linear combination of vectors in $\mathbb{R}^5$.**

[3] (b) **Find a particular solution of the nonhomogeneous system $A\boldsymbol{x} = \boldsymbol{b}$.**

The solution should be in terms of b_1, b_2, b_3, but contain no arbitrary constants.

ANSWER BOX

We convert the augmented matrix to reduced row-echelon form by means of row operations:

$$\left[\begin{array}{ccccc|c} 1 & -1 & -2 & -3 & -4 & b_2 \\ 0 & 0 & -1 & -2 & -3 & b_3 \\ 0 & 0 & -3 & -4 & -5 & b_1 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -1 & -2 & -3 & -4 & b_2 \\ 0 & 0 & 1 & 2 & 3 & -b_3 \\ 0 & 0 & -3 & -4 & -5 & b_1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 2 & b_2 - 2b_3 \\ 0 & 0 & 1 & 2 & 3 & -b_3 \\ 0 & 0 & 0 & 2 & 4 & b_1 - 3b_3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & (b_2 - 2b_3) - \frac{1}{2}(b_1 - 3b_3) \\ 0 & 0 & 1 & 0 & -1 & -b_1 + 2b_3 \\ 0 & 0 & 0 & 1 & 2 & \frac{1}{2}(b_1 - 3b_3) \end{array}\right]$$

From the reduced augmented matrix we can read off the required information.

(a) The general solution of the homogeneous system is

$$\boldsymbol{x} = r_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \qquad (r_1, r_2 \in \mathbb{R}).$$

(b) We get a particular solution by setting $x_2 = x_5 = 0$

$$\boldsymbol{x} = \left[\frac{1}{2}(b_1 + 2b_2 - b_3), 0, -b_1 + 2b_3, \frac{1}{2}(b_1 - 3b_3), 0\right].$$

■

[4] **3.** Let A denote the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Express A as a product of elementary matrices.

ANSWER BOX

By row operations we convert A to I:

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \qquad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad R_1 \to R_1 + R_2,\ R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 \to (-1)R_2,\ R_3 \to (1/2)R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1 \to R_1 - R_3\,.$$

Now A is equal to the following product:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From left to right the terms of the product are the elementary matrices corresponding to the inverses of the seven row operations used to convert A to I. ■

4. Let $A \in \mathbb{R}^{5\times 6}$ have columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_6$ respectively. Let the reduced row-echelon form of A be

$$H = \begin{bmatrix} 1 & 0 & -2 & 0 & -3 & -4 \\ 0 & 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

[2] (a) **Write down a basis for the row space of A.**

ANSWER BOX

$$\{[1,0,-2,0,-3,-4],\ [0,1,-1,0,-2,-3],\ [0,0,0,1,-1,-2]\}$$

■

[2] (b) **Write down a basis for the column space of A.**

ANSWER BOX

Since the pivots of A fall in columns 1, 2, and 4, the corresponding columns of A form a basis for the column space of A

$$\{\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_4\}\ .$$

■

[2] (c) **Write down a basis for the nullspace of A.**

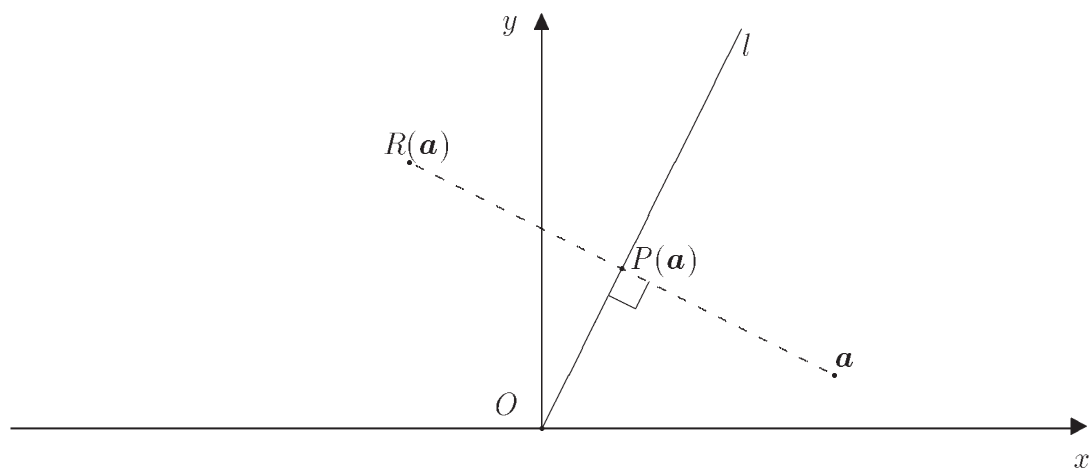
ANSWER BOX

$$\{[2,1,1,0,0,0],\ [3,2,0,1,1,0],\ [4,3,0,2,0,1]\}\ .$$

■

5. Let l be the line in $\mathbb{R}^2$ whose equation is $y = 2x$. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $P(\boldsymbol{a})$ is the projection of $\boldsymbol{a}$ on l.

Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the function such that $R(\boldsymbol{a})$ is the reflection of $\boldsymbol{a}$ in l.



[3] (a) **Show that P is a linear transformation and find the matrix which is its standard representation.**

ANSWER BOX

We need to find a formula for $P(\boldsymbol{a})$. One way of doing this is to observe that $P(\boldsymbol{a})$ has the form $[r, 2r]$ and that $(P(\boldsymbol{a}) - \boldsymbol{a}) \cdot [1, 2] = 0$. We easily deduce that

$$P([a_1, a_2]) = \left[\frac{a_1 + 2a_2}{5}, \frac{2a_1 + 4a_2}{5}\right] = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}.$$

Since P is represented by a matrix, P preserves addition and scalar multiplication. Therefore P is a linear transformation. ■

[3] (b) **Express R as a linear combination of P and I the identity transformation on $\mathbb{R}^2$.**

Hence show that R is also a linear transformation.

ANSWER BOX

From the definition of reflection we see that $R(\boldsymbol{a}) - P(\boldsymbol{a}) = P(\boldsymbol{a}) - \boldsymbol{a}$ for all $\boldsymbol{a} \in \mathbb{R}^2$. Therefore in the space of functions from $\mathbb{R}^2$ into $\mathbb{R}^2$ we have

$$R = 2P - I\,.$$

Both P and I are linear transformations. Therefore R, being a linear combination of linear transformations, is also a linear transformation. ■

[4] **6.** (a) Let V be a vector space over $\mathbb{R}$ and S be a subset of V.

State a criterion for S to be a subspace of V.

ANSWER BOX

S is a subspace if it satisfies three conditions:

1. $S \neq \emptyset$
2. $\boldsymbol{u} + \boldsymbol{v} \in S$ for all $\boldsymbol{u}, \boldsymbol{v} \in S$
3. $r\boldsymbol{u} \in S$ for all $r \in \mathbb{R}$ and all $\boldsymbol{u} \in S$.

■

[4] (b) **Which of the following sets are subspaces of $\mathbb{R}^3$?**

1. $S = \{[x, y, z] : x \geq 0\}$
2. $S = \{[x, y, z] : x + 3y = z\}$
3. $S = \{[x, y, z] : xy = 0\}$

Justify your answers briefly.

ANSWER BOX

1. S is not a subspace because $[1, 0, 0] \in S$ but $(-1)[1, 0, 0] \notin S$. ■

2. S is a subspace, the nullspace of $[1\ 3\ \ -1]$.
   In more detail, $S \neq \emptyset$ since $\mathbf{0} \in S$. Let $\boldsymbol{u} = [u_1, u_2, u_3]$, $\boldsymbol{v} = [v_1, v_2, v_3]$ be in S. Then
   $$(u_1 + v_1) + 3(u_2 + v_2) = (u_1 + 3u_2) + (v_1 + 3v_2) = u_3 + v_3\,.$$
   Therefore $\boldsymbol{u} + \boldsymbol{v} \in S$. So S is closed under addition. Similarly, S is closed under scalar multiplication. ■

3. S is not a subspace because $[1, 0, 0]$, $[0, 1, 0] \in S$ but $[1, 0, 0] + [0, 1, 0] \notin S$. ■

7. Let $V = \mathbb{R}^{2\times 2}$ denote the vector space over $\mathbb{R}$ whose vectors are the 2×2 matrices with entries from $\mathbb{R}$. Let

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{v}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

[4] (a) **Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly independent in V.**

ANSWER BOX

Suppose that $\boldsymbol{c} = [c_1, c_2, c_3, c_4] \in \mathbb{R}^4$ satisfies $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \boldsymbol{0}$. Then

$$\begin{bmatrix} c_1 + c_2 + c_3 & c_1 + c_4 \\ c_3 + c_4 & c_2 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$3c_1 = (c_1 + c_2 + c_3) + 2(c_1 + c_4) - (c_3 + c_4) - (c_2 + c_4) = 0\,.$$

Hence $c_1 = 0$. Since $c_1 + c_4 = 0$, $c_4 = 0$. Since $c_3 + c_4 = c_2 + c_4 = 0$, it also follows that $c_2 = c_3 = 0$. This is enough. ■

[3] (b) **Find the coordinate vector of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with respect to the ordered basis $\mathcal{B} = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \rangle$.**

ANSWER BOX

Let $\boldsymbol{u}$ denote $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $\boldsymbol{w}$ denote $\boldsymbol{u} + \boldsymbol{v}_4$ the vector we are interested in. Observe that

$$3\boldsymbol{u} = (\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) - \boldsymbol{v}_4\,.$$

Therefore

$$\boldsymbol{w} = \frac{1}{3}\left[(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) - \boldsymbol{v}_4\right] + \boldsymbol{v}_4 = \frac{1}{3}\boldsymbol{v}_1 + \frac{1}{3}\boldsymbol{v}_2 + \frac{1}{3}\boldsymbol{v}_3 + \frac{2}{3}\boldsymbol{v}_4\,.$$

Therefore $\boldsymbol{w}_{\mathcal{B}} = [1/3, 1/3, 1/3, 2/3]$. ■

[4] **8.** (a) **Evaluate the determinant**

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix}$$

ANSWER BOX

Expanding by the last row and then by the second row we get

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 8 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 5 & 6 & 7 \end{vmatrix} = -8 \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = 32.$$

■

[3] (b) Let A be an $n \times n$ matrix and A_{ij} denote the i, j-th minor of A.

State the formula which expands det(A) by the i-th row.

ANSWER BOX

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) .$$

■

9. Let A denote the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

[4] (a) **Find the eigenvalues of A.**

[4] (b) **Discover whether A is diagonalizable and explain your answer carefully.**

ANSWER BOX

(a) We find the characteristic polynomial simultaneously finding its factors:

$$\begin{aligned}
|\lambda I - A| &= \begin{vmatrix} \lambda-3 & -1 & 1 \\ 1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} \\
&= \begin{vmatrix} \lambda-2 & -1 & 0 \\ 2-\lambda & \lambda-1 & \lambda-2 \\ 0 & -1 & \lambda-2 \end{vmatrix} && C_1 \to C_1 - C_2,\ C_3 \to C_3 + C_2 \\
&= (\lambda-2)^2 \begin{vmatrix} 1 & -1 & 0 \\ -1 & \lambda-1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
&= (\lambda-2)^2 \begin{vmatrix} 1 & 0 & 0 \\ -1 & \lambda-1 & 1 \\ 0 & 0 & 1 \end{vmatrix} && C_2 \to C_2 + C_1,\ C_2 \to C_2 + C_3 \\
&= (\lambda-2)^2(\lambda-1)\,.
\end{aligned}$$

The eigenvalues are $\lambda = 2$ (algebraic multiplicity 2), and $\lambda = 1$. ■

(b) In the usual way we find the eigenspaces

$$\begin{aligned}
E_1 &= \mathsf{sp}([1,-1,1]) \\
E_2 &= \mathsf{sp}([1,0,1],[0,1,1])\,.
\end{aligned}$$

Since there is a basis of $\mathbb{R}^3$, namely

$$\{[1,-1,1],[1,0,1],[0,1,1]\}, \qquad (1)$$

consisting of eigenvectors of A, A is diagonalizable. A diagonalizing matrix C is obtained by taking as columns the vectors in (1). ■

[3] **10.** (a) **Find a basis for the orthogonal complement in $\mathbb{R}^4$ of the space $W = \mathbf{sp}([1,2,-1,1],[1,-1,1,1])$.**

[4] (b) Let $\boldsymbol{b} = [0,3,3,3]$.

Find the projection $\boldsymbol{b}_W$ of $\boldsymbol{b}$ on W.

ANSWER BOX

(a) We have

$$W^\perp = \text{nullspace}\left(\begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}\right).$$

Since

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{-2}{3} & 0 \end{bmatrix},$$

we see that $W^\perp = \text{sp}([-1,2,3,0],[-1,0,0,1])$. By inspection the spanning vectors we have given for $W^\perp$ are are linearly independent and hence a basis. ■

(b) Let $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ be the basis vectors given for W. Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ be the basis vectors found for $W^\perp$. Let all these vectors be seen as column vectors. By row-reduction

$$\left[\begin{array}{cccc|c} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{b} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & & & & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \end{array}\right].$$

Therefore $\boldsymbol{b} = \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{v}_1 + \boldsymbol{v}_2$. It follows that

$$\boldsymbol{b}_W = \boldsymbol{w}_1 + \boldsymbol{w}_2 = [2,1,0,2].$$

■

Note. There are at least two other methods one might use here: (1) finding an orthogonal basis for W and then summing the projections of $\boldsymbol{b}$ along the basis vectors, (2) using the projection matrix P for W.

11. Let $W = \mathrm{sp}([1,-1,-1,1],[2,2,1,1],[0,1,0,1])$.

[5] (a) **Find an orthogonal basis for W.**

[3] (b) **Find an orthonormal basis for W.**

ANSWER BOX

(a) Let the given basis vectors be denoted $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ respectively. By inspection, $\boldsymbol{v}_1$ is orthogonal to both $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$. So it is sufficient to find an orthogonal basis for sp$(\boldsymbol{v}_2, \boldsymbol{v}_3)$. By the Gram-Schmidt method such a basis is

$$\boldsymbol{v}_3, \quad \boldsymbol{v}_2 - \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_3}{\boldsymbol{v}_3 \cdot \boldsymbol{v}_3}\right)\boldsymbol{v}_3\,.$$

Thus a suitable basis for W is

$$\{[1,-1,-1,1],[0,1,0,1],[2,1/2,1,-1/2]\}\,.$$

■

(b) We obtain an orthonormal basis by normalizing the basis above:

$$\{(1/2)[1,-1,-1,1],(1/\sqrt{2})[0,1,0,1],(1/\sqrt{22})[4,1,2,-1]\}\,.$$

■

12. Let $\boldsymbol{b}$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be nonzero vectors in $\mathbb{R}^n$ such that

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0 \qquad (1 \le i < j \le k).$$

Let W denote $\mathrm{sp}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$.

Let A denote the $n \times k$ matrix $[\boldsymbol{v}_1 \ \ldots \ \boldsymbol{v}_k]$ whose columns are $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$.

[4] (a) **Prove that the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ is linearly independent.**

[4] (b) **Prove that**

$$\boldsymbol{b}_W = A\left(A^T A\right)^{-1} A^T \boldsymbol{b}.$$

What has to be shown here is that the vector $\boldsymbol{b}_W$ defined by this formula satisfies $\boldsymbol{b}_W \in W$ and $\boldsymbol{b} - \boldsymbol{b}_W \perp W$.

ANSWER BOX

(a) Suppose that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_k\boldsymbol{v}_k = \boldsymbol{0}.$$

Taking the dot product with $\boldsymbol{v}_i$ we get

$$c_1(\boldsymbol{v}_1 \cdot \boldsymbol{v}_i) + c_2(\boldsymbol{v}_2 \cdot \boldsymbol{v}_i) + \ldots + c_k(\boldsymbol{v}_k \cdot \boldsymbol{v}_i) = 0.$$

Since $\boldsymbol{v}_j \cdot \boldsymbol{v}_i = 0$ except when $j = i$, we get $c_i(\boldsymbol{v}_i \cdot \boldsymbol{v}_i) = 0$. Since $\boldsymbol{v}_i \neq \boldsymbol{0}$, we have $\boldsymbol{v}_i \cdot \boldsymbol{v}_i \neq 0$. Hence $c_i = 0$. But i was arbitrary. Thus $c_i = 0$ for all i, $1 \le i \le k$. This is enough. ∎

(b) First note that, since $\boldsymbol{b}_W$ has the form $A\boldsymbol{c}$, $\boldsymbol{b}_W$ is a linear combination of the columns of A. Therefore $\boldsymbol{b}_W \in W$.

Next, observe that

$$\begin{aligned} A^T(\boldsymbol{b} - \boldsymbol{b}_W) &= A^T\boldsymbol{b} - A^T\boldsymbol{b}_W \\ &= A^T\boldsymbol{b} - A^T(A\left(A^T A\right)^{-1} A^T\boldsymbol{b}) \\ &= A^T\boldsymbol{b} - (A^T A)\left(A^T A\right)^{-1}(A^T\boldsymbol{b}) \\ &= A^T\boldsymbol{b} - A^T\boldsymbol{b} \ = \ \boldsymbol{0}. \end{aligned}$$

Therefore $\boldsymbol{b} - \boldsymbol{b}_W$ is orthogonal to every row of A^T. Hence $\boldsymbol{b} - \boldsymbol{b}_W$ is orthogonal to all of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. So $\boldsymbol{b} - \boldsymbol{b}_W \perp W$. ∎

13. Let V denote the vector space over $\mathbb{R}$ consisting of all polynomials in $\mathbb{R}[x]$ of degree at most 2.
Let

$$\mathcal{B} = \langle x+1, (x+1)^2, 1\rangle, \quad \mathcal{B}' = \langle 2x-1, 2x+1, x^2+x\rangle.$$

Let $F : V \to V$ be the unique linear transformation such that $F(1) = 1+x$, $F(x) = x + x^2$, $F(x^2) = 1 + x^2$

[4] (a) **Find a matrix $C \in \mathbb{R}^{3\times 3}$ such that, for all $\boldsymbol{v}$ in V,**

$$C\boldsymbol{v}_{\mathcal{B}} = \boldsymbol{v}_{\mathcal{B}'}$$

[4] (b) **Find the matrix $[F]_{\mathcal{B}',\mathcal{B}'}$ which represents F with respect to $\mathcal{B}'$, $\mathcal{B}'$.**

ANSWER BOX

(a) We need the following calculations:

$$\boldsymbol{b}_1 = x+1 = -\frac{1}{4}\boldsymbol{b}'_1 + \frac{3}{4}\boldsymbol{b}'_2$$

$$\boldsymbol{b}_2 = (x+1)^2 = 1 + 2x + x^2 = -\frac{1}{4}\boldsymbol{b}'_1 + \frac{3}{4}\boldsymbol{b}'_2 + \boldsymbol{b}'_3$$

$$\boldsymbol{b}_3 = 1 = -\frac{1}{2}\boldsymbol{b}'_1 + \frac{1}{2}\boldsymbol{b}'_2$$

So the desired matrix C is $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. ■

(b) Similarly, we have

$$F(\boldsymbol{b}'_1) = F(2x-1) = -1 + x + 2x^2 = \frac{1}{4}\boldsymbol{b}'_1 - \frac{3}{4}\boldsymbol{b}'_2 + 2\boldsymbol{b}'_3$$

$$F(\boldsymbol{b}'_2) = F(2x+1) = 1 + 3x + 2x^2 = -\frac{1}{4}\boldsymbol{b}'_1 + \frac{3}{4}\boldsymbol{b}'_2 + 2\boldsymbol{b}'_3$$

$$F(\boldsymbol{b}'_3) = F(x^2+x) = 1 + x + 2x^2 = -\frac{3}{4}\boldsymbol{b}'_1 + \frac{1}{4}\boldsymbol{b}'_2 + 2\boldsymbol{b}'_3$$

Therefore the matrix representing F with respect to $\mathcal{B}'$ is

$$\begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ 2 & 2 & 2 \end{bmatrix}.$$

■

[9] **14.** Consider the surface S in $\mathbb{R}^3$ whose equation is

$$-x^2 + y^2 - z^2 - 2yz - 6zx + 2xy = 1\,.$$

Determine as well as you can the nature of the surface S.

Some questions that should be addressed are:

Does S have a centre? Is S connected? Are there axes?

Make it clear how you reach your conclusions.

ANSWER BOX

The symmetric matrix of coefficients of the quadratic form on the left of the equation is

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & -1 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{vmatrix} \lambda+1 & -1 & 3 \\ -1 & \lambda-1 & 1 \\ 3 & 1 & \lambda+1 \end{vmatrix} = \lambda(\lambda-3)(\lambda+4)\,.$$

Because A is symmetric we know that there exists an orthogonal matrix C such that the substitution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix}$$

converts the given equation to

$$3\overline{x}^2 - 4\overline{y}^2 = 1\,.$$

This is a hyperbolic cylinder; it has two components. There is no unique centre, but the centre of the hyperbola in each plane parallel to the $\overline{x}\,\overline{y}$-plane is a point of reflection for the surface. Similarly, for the axes: we get two in each plane parallel to the $\overline{xy}$-plane. The $\overline{z}$-axis is also an axis of the surface, the only one in its parallellism class. ■