MATH 232 Key for sample final exam, August 1998

[4] **1.** (a) **Define the term "reduced row-echelon matrix".**

ANSWER BOX

A matrix is *reduced row-echelon* if the following conditions are satisfied

1. every zero row lies below every nonzero row
2. all pivots are equal to 1
3. each pivot lies to the left of all pivots of rows below it
4. every other entry in the column of a pivot is equal to 0. ■

[3] (b) **Find a reduced row-echelon matrix row-equivalent to**

$$\begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 2 \\ 4 & -4 & 4 \end{bmatrix}$$

ANSWER BOX

Applying the row operations $R_4 \to R_4 - 2R_1$, $R_1 \to R_1 + 2R_1$, we get

$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & 2 & 2 \\ 0 & 2 & 6 \end{bmatrix}.$$

Applying the row operations $R_3 \to R_3 - 2R_1$, $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, applying $R_1 \leftrightarrow R_2$, $R_1 \to (-1)R_1$, we get

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

■

2. Let

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

[4] (a) Express the general solution of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ as a linear combination of vectors in $\mathbb{R}^5$.

ANSWER BOX

By performing on $[A|\boldsymbol{b}]$ the row operations $R_1 \to R_1 + R_2$, $R_2 \to R_2 + 2R_3$, $R_1 \to R_1 - R_3$, and $R_1 \to (1/2)R_1$ we get

$$[H|\boldsymbol{c}] = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

From this we see that the general solution of the homogeneous system is

$$\boldsymbol{x} = r_1[-2, -3, 1, 0, 0] + r_2[0, -2, 0, -1, 1] \qquad (r_1,\, r_2 \in \mathbb{R}).$$

■

[3] (b) Write down the general solution of the nonhomogeneous system $A\boldsymbol{x} = \boldsymbol{b}$.

ANSWER BOX

From the reduced augmented matrix, we see that

$$\boldsymbol{x} = [0, 2, 0, 1, 0] + r_1[-2, -3, 1, 0, 0] + r_2[0, -2, 0, -1, 1] \qquad (r_1,\, r_2 \in \mathbb{R}).$$

■

3. The matrix A in $\mathbb{R}^{n\times n}$ is defined to be *invertible* if there exists B in $\mathbb{R}^{n\times n}$ such that

$$AB = BA = I\,.$$

Let A, $C \in \mathbb{R}^{n\times n}$ both be invertible.

[3] (a) **Show that AC is invertible.**

ANSWER BOX

Since A and C are invertible there exist $B, D \in \mathbb{R}^{n\times n}$ such that

$$AB = BA = I \text{ and } CD = DC = I\,.$$

Using the associativity of matrix multiplication, we have

$$\begin{aligned}(AC)(DB) &= A(CD)B = AIB = AB = I\\ (DB)(AC) &= D(BA)C = DIC = DC = I\,.\end{aligned}$$

Thus DB witnesses that AC is invertible. ■

[3] (b) **Show that A^T is invertible.**

ANSWER BOX

Let B witness that A is invertible, i.e.,

$$AB = BA = I\,.$$

Taking the transpose of each term we have

$$B^T A^T = A^T B^T = I\,.$$

Therefore B^T witnesses that A^T is invertible. ■

[3] **4.** (a) **Explain how to compute the rank of a matrix.**

ANSWER BOX

By performing row operations on the given matrix A find a reduced row-echelon matrix H row-equivalent to A. The rank of A is the number of nonzero rows of H. ■

[4] (b) **Explain why it is true that**

$$\mathrm{rank}(AB) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$$

for all matrices A, B such that AB is defined.

ANSWER BOX

The crux of the matter is that the columns of AB are linear combinations of the columns of A — indeed, if $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are the columns of A and $[b_{1j}, \ldots, b_{nj}]$ is the j-th column of B, then the j-th column of AB is $b_{1j}\boldsymbol{a}_1 + \ldots + b_{nj}\boldsymbol{a}_n$. Therefore colspace$(AB) \subseteq$ colspace(A) and so

$$\mathsf{rank}(AB) = \dim(\mathsf{colspace}(AB)) \leq \dim(\mathsf{colspace}(A)) = \mathsf{rank}(A)\,.$$

Applying this result to the product $B^T A^T$ we obtain

$$\mathsf{rank}(AB) = \mathsf{rank}\left((AB)^T\right) = \mathsf{rank}(B^T A^T) \leq \mathsf{rank}(B^T) = \mathsf{rank}(B)\,.$$

This is enough. ■

[6] **5. Find a formula which defines a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies**

$$F([1,1,1]) = [0,1,1], \qquad F([1,-1,-1]) = [1,0,0] .$$

ANSWER BOX

The requirement above specifies F on a subspace W of dimension 2. To complete the specification of F on $\mathbb{R}^3$ we require that $F([0,1,0]) = [0,0,0]$. Since F is linear,

$$F([1,0,0]) = F\left((1/2)[[1,1,1] + (1/2)[[1,-1,-1]\right) = (1/2)\left([0,1,1] + [1,0,0]\right)$$
$$F([0,0,1]) = F([1,1,1] - [1,0,0] - [0,1,0]) = [0,1,1] - [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}] - [0,0,0] .$$

Thus the standard matrix representation of F is

$$[F(\boldsymbol{e}_1)\ F(\boldsymbol{e}_2)\ F(\boldsymbol{e}_3)] = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} .$$

Therefore a formula defining F is

$$F([x,y,z]) = (1/2)\left[x - z, x + z, x + z\right] .$$

■

Definition of vector space

A *vector space over* $\mathbb{R}$ is a set V of vectors together with a distinguished vector $\mathbf{0}$ in V and three functions

$$(\boldsymbol{u},\boldsymbol{v}) \mapsto \boldsymbol{u}+\boldsymbol{v}, \quad \boldsymbol{v} \mapsto -\boldsymbol{v}, \quad (r,\boldsymbol{v}) \mapsto r\boldsymbol{v} \qquad (\boldsymbol{u},\, \boldsymbol{v} \in V,\, r \in \mathbb{R})$$

which satisfy

A1	$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$	associative law
A2	$\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$	commutative law
A3	$\mathbf{0}+\boldsymbol{u} = \boldsymbol{u}$	additive identity
A4	$\boldsymbol{u}+(-\boldsymbol{u}) = \mathbf{0}$	additive inverse
S1	$r(\boldsymbol{u}+\boldsymbol{v}) = r\boldsymbol{u}+r\boldsymbol{v}$	distributivity
S2	$(r+s)\boldsymbol{u} = r\boldsymbol{u}+s\boldsymbol{u}$	distributivity
S3	$r(s\boldsymbol{u}) = (rs)\boldsymbol{u}$	associative law
S4	$1\boldsymbol{u} = \boldsymbol{u}$	scale preservation

for all $\boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{w} \in V$ and $r,\, s \in \mathbb{R}$.

[6] **6. From the axioms for a vector space show that for all vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in V,**

$$\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{u}+\boldsymbol{w} \;\Rightarrow\; \boldsymbol{v} = \boldsymbol{w}\,.$$

ANSWER BOX

Suppose that $\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{u}+\boldsymbol{w}$. Then

$$\begin{aligned}
\boldsymbol{v} &= \mathbf{0}+\boldsymbol{v} && \text{by A3}\\
&= (\boldsymbol{u}+(-\boldsymbol{u}))+\boldsymbol{v} && \text{by A4}\\
&= (\boldsymbol{u}+\boldsymbol{v})+(-\boldsymbol{u}) && \text{by A1, A2}\\
&= (\boldsymbol{u}+\boldsymbol{w})+(-\boldsymbol{u}) && \text{hypothesis}\\
&= (\boldsymbol{u}+(-\boldsymbol{u}))+\boldsymbol{w} && \text{by A1, A2}\\
&= \mathbf{0}+\boldsymbol{w} && \text{by A4}\\
&= \boldsymbol{w} && \text{A3}\,.
\end{aligned}$$

This is enough. ■

7. Let $V = \mathbb{R}^{2\times 2}$ denote the vector space over $\mathbb{R}$ whose vectors are the 2×2 matrices with entries from $\mathbb{R}$. Let

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ \boldsymbol{v}_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

[3] (a) **Show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is linearly dependent.**

ANSWER BOX

By inspection,

$$\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_3 + \boldsymbol{v}_4 \,.$$

■

[3] (b) **Find $\boldsymbol{u} \in V$ such that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{u}\}$ is a basis for V.**

ANSWER BOX

We can take $\boldsymbol{u} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\boldsymbol{u}$, $\boldsymbol{v}_1 - \boldsymbol{u}$, $\boldsymbol{v}_3 - \boldsymbol{u}$, $\boldsymbol{v}_2 - \boldsymbol{v}_3 + \boldsymbol{u}$ are the four 2×2 matrices with one entry equal to 1 and all the other entries equal to 0.

Thus $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{u}\}$ is a basis for V. ■

[4] **8.** (a) **Evaluate the determinant**

$$\begin{vmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 & 8 \\ 0 & 9 & 0 & 10 & 0 \\ 11 & 0 & 12 & 0 & 13 \end{vmatrix}$$

ANSWER BOX

Performing the row operations $R_5 \to R_5 - R_3$ and $R_3 \to R_3 - R_1$ yields a matrix in which rows 3 and 5 are the same. These row operations do not change the value of the determinant.

So the given determinant is 0. ■

[3] (b) Let A be a square matrix.

State the relationship between det(A) and rank(A).

ANSWER BOX

Let A be in $\mathbb{R}^{n\times n}$. Then

$$\det(A) \neq 0 \iff \operatorname{rank}(A) = n\,.$$

9. Let A denote the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

[4] (a) **Find the eigenvalues of A.**

ANSWER BOX

The characteristic polynomial is

$$\begin{vmatrix} \lambda-1 & 0 & 0 & -1 \\ 0 & \lambda-1 & 0 & -1 \\ -1 & 0 & \lambda & -1 \\ -1 & -1 & -1 & \lambda-1 \end{vmatrix} = \begin{vmatrix} \lambda & 0 & -\lambda & 0 \\ 0 & \lambda-1 & 0 & -1 \\ -1 & 0 & \lambda & -1 \\ -1 & -1 & -1 & \lambda-1 \end{vmatrix} = \lambda \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & \lambda-1 & 0 & -1 \\ -1 & 0 & \lambda & -1 \\ -1 & -1 & -1 & \lambda-1 \end{vmatrix} =$$

$$\lambda \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & \lambda-1 & 0 & -1 \\ 0 & 0 & \lambda-1 & -1 \\ 0 & -1 & -2 & \lambda-1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & -1 \\ -1 & -2 & \lambda-1 \end{vmatrix} = \lambda \begin{vmatrix} \lambda-1 & -2(\lambda-1) & -1 \\ 0 & \lambda-1 & -1 \\ -1 & 0 & \lambda-1 \end{vmatrix} =$$

$$\lambda(\lambda-1) \begin{vmatrix} \lambda-1 & -2 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1) \begin{vmatrix} \lambda-1 & 0 & -3 \\ 0 & 1 & -1 \\ -1 & 0 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1)\left((\lambda-1)^2-3\right).$$

Therefore the eigenvalues are $\lambda = 0,\ 1,\ 1 \pm \sqrt{3}$. ■

[4] (b) **Find a matrix C such that $C^{-1}AC$ is a diagonal.**

ANSWER BOX

By inspection eigenvectors belonging to 0, 1 are $[1, 1, -1, -1]$, $[1, -2, 1, 0]$ respectively. We compute an eigenvector belonging to $1 + \sqrt{3}$ by

$$\text{nullspace}\left((1+\sqrt{3})I - A\right) = \text{nullspace}\left(\begin{bmatrix} \sqrt{3} & 0 & 0 & -1 \\ 0 & \sqrt{3} & 0 & -1 \\ -1 & 0 & 1+\sqrt{3} & -1 \\ -1 & -1 & -1 & \sqrt{3} \end{bmatrix}\right) = \text{sp}([1, 1, 1, \sqrt{3}]).$$

Hence $[1, 1, 1, \sqrt{3}]$ is an eigenvector belonging to $1 + \sqrt{3}$, and similarly $[1, 1, 1, -\sqrt{3}]$ is an eigenvector belonging to $1 - \sqrt{3}$. Therefore we may take

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 0 & \sqrt{3} & -\sqrt{3} \end{bmatrix}$$

■

[3] **10.** (a) **Find the projection of** $[2, -1, 3]$ **on** $\mathbf{sp}([1, 2, -1])$.

ANSWER BOX

$$\boldsymbol{p} = \left(\frac{[2, -1, 3] \cdot [1, 2, -1]}{[1, 2, -1] \cdot [1, 2, -1]} \right) [1, 2, -1] = (-3/6)[1, 2, -1] = \frac{1}{2}[1, 2, -1] \, .$$

[4] (b) **Find a formula for the projection of** $\boldsymbol{b} = [b_1, b_2, b_3]$ **on the subspace** $\mathbf{sp}([1, 1, -1], [-1, 1, 1])$.

ANSWER BOX

Let A denote the matrix $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}^T$. The projection matrix for the subspace $W = \mathrm{sp}([1, 1, -1], [-1, 1, 1])$ is

$$P = A(A^T A)^{-1} A^T = A \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} A^T = A \left(\frac{1}{8} \right) \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} A^T$$

$$= \left(\frac{1}{8} \right) A \begin{bmatrix} 2 & 4 & -2 \\ -2 & 4 & 2 \end{bmatrix} = \left(\frac{1}{8} \right) \begin{bmatrix} 4 & 0 & -4 \\ 0 & 8 & 0 \\ -4 & 0 & 4 \end{bmatrix} .$$

So the projection of $\boldsymbol{b}$ on W is $P\boldsymbol{b} = \left[\frac{b_1 - b_3}{2}, b_2, \frac{b_3 - b_1}{2} \right]$. ■

[3] **11.** (a) **State three conditions on a matrix $A \in \mathbb{R}^{n\times n}$ which are equivalent to A being an orthogonal matrix.**

ANSWER BOX

Any three of the following five conditions are acceptable:

1. The columns of A form an orthonormal basis of $\mathbb{R}^n$.
2. The rows of A form an orthonormal basis of $\mathbb{R}^n$.
3. $A^T A = I$.
4. $\|A\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
5. $(A\boldsymbol{x}) \cdot (A\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

[3] (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal linear transformation such that

$$T([1,0]) = [1/2, \sqrt{3}/2].$$

Explain why there are only two possibilities for T and describe them.

ANSWER BOX

The standard matrix representation of T is an orthogonal matrix and has columns $T([1,0])$ and $T([0,1])$.

From condition 1 we see that $T([1,0])$ and $T([0,1])$ form an orthonormal basis of $\mathbb{R}^2$. Since $T([1,0]) = [1/2, \sqrt{3}/2]$, we see that, if $T([0,1]) = [a_1, a_2]$, then ${a_1}^2 + {a_2}^2 = 1$ and $(1/2)a_1 + (\sqrt{3}/2)a_2 = 0$.

Therefore $T([0,1])$ is either $[\sqrt{3}/2, -1/2]$ or $[-\sqrt{3}/2, 1/2]$. ■

[8] **12.** The following data points are given:

$$(-2,-8),\ (-1,-8),\ (2,0),\ (3,0),\ (4,2),\ (6,8)$$

By using a method from linear algebra find the least-squares linear fit for these data points.

Your answer should make it clear what method you are using.

ANSWER BOX

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 2 & 3 & 4 & 6 \end{bmatrix}^T$ and $\boldsymbol{b} = \begin{bmatrix} -8 & -8 & 0 & 0 & 2 & 8 \end{bmatrix}^T$. Then

$$\begin{aligned} (A^TA)^{-1}A^T &= \begin{bmatrix} 6 & 12 \\ 12 & 70 \end{bmatrix}^{-1} = \frac{1}{276}\begin{bmatrix} 70 & -12 \\ -12 & 6 \end{bmatrix} A^T \\ &= \frac{1}{276}\begin{bmatrix} 94 & 82 & 46 & 34 & 22 & -2 \\ -24 & -18 & 0 & 6 & 12 & 24 \end{bmatrix} \end{aligned}$$

According to the theory, the coefficients of the least-squares linear fit are given by

$$\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = (A^TA)^{-1}A^T\boldsymbol{b} = \begin{bmatrix} 1380/276 \\ 552/276 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

So the least-squares linear fit is $y = 5 + 2x$. ■

13. Let ${}^{\mathbb{R}}\mathbb{R}$ denote the vector space over $\mathbb{R}$ consisting of all functions $f : \mathbb{R} \to \mathbb{R}$. Let V denote the subspace of ${}^{\mathbb{R}}\mathbb{R}$. spanned by $\{1, \sin 2x, \cos 2x\}$. Let

$$\mathcal{B} = \langle 1, \sin 2x, \cos 2x \rangle, \quad \mathcal{B}' = \langle \sin^2 x, \cos^2 x, \sin x \cos x \rangle.$$

Let $F : V \to V$ be the unique linear transformation which maps $\mathcal{B}$ to $\mathcal{B}'$ in the sense that $F(1) = \sin^2 x$, $F(\sin 2x) = \cos^2 x$, and $F(\cos 2x) = \sin x \cos x$.

[4] (a) **Find a matrix $C \in \mathbb{R}^{3\times 3}$ such that, for all $\boldsymbol{v}$ in V,**

$$C\boldsymbol{v}_{\mathcal{B}} = \boldsymbol{v}_{\mathcal{B}'}$$

ANSWER BOX

Note that

$$\begin{aligned} 1 &= \sin^2 x + \cos^2 x \\ \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= -\sin^2 x + \cos^2 x\,. \end{aligned}$$

Thus the required matrix C is

$$C_{\mathcal{B},\mathcal{B}'} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

∎

[4] (b) **Find the matrix $[F]_{\mathcal{B},\mathcal{B}}$ which represents F with respect to $\mathcal{B}$, $\mathcal{B}$.**

ANSWER BOX

Notice that the matrix of F with respect to $\mathcal{B}$, $\mathcal{B}'$ is I the 3×3 identity matrix. Therefore the matrix of F with respect to $\mathcal{B}$, $\mathcal{B}$ is

$$\begin{aligned} [F]_{\mathcal{B},\mathcal{B}} &= C_{\mathcal{B}',\mathcal{B}}\,[F]_{\mathcal{B},\mathcal{B}'} = (C_{\mathcal{B},\mathcal{B}'})^{-1} \\ &= -\frac{1}{4}\begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}. \end{aligned}$$

∎

[5] **14.** Consider the curve in $\mathbb{R}^2$ whose equation is

$$6x^2 + \sqrt{24}xy + 7y^2 = 1\,.$$

Show that this curve is an ellipse and find the length of its major and minor axes.

ANSWER BOX
The symmetric matrix of coefficients of the quadratic form $6x^2 + \sqrt{24}xy + 7y^2$ is

$$A = \begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 7 \end{bmatrix}.$$

The characteristic polynomial is $(\lambda - 9)(\lambda - 4)$. Letting C be the orthogonal diagonalizing matrix, the change of variables $\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$ converts the equation to

$$9\overline{x}^2 + 4\overline{y}^2 = 1\,.$$

This is clearly the equation of an ellipse. The major axis has length 1, while the minor axis has length $2/3$. ■

[6] **15.** Consider the surface S in $\mathbb{R}^3$ whose equation is

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 3x + z = 1\,.$$

Show that S is cylindrical in the sense that there is a unit vector $\boldsymbol{u}$ such that S is invariant under translation by any scalar multiple of $\boldsymbol{u}$.

ANSWER BOX

Here the symmetric matrix of coefficients has eigenvalues 3, 0, where 0 has multiplicity 2. So the associated orthogonal transformation converts the equation to the form

$$3\overline{x}^2 - \frac{2}{\sqrt{3}}\overline{x} + a\overline{y} + b\overline{z} = 1\,.$$

The exact values of a and b do not matter except that one of them is nonzero. Making a rotation about the $\overline{x}$-axis we can convert the equation to the form

$$3\overline{x}^2 - \frac{2}{\sqrt{3}}\overline{x} + c\overline{y} = 1\,.$$

This is enough. The unit vector $\boldsymbol{u}$ is the one in the direction of the z-axis. ■