DEPARTMENT OF MATHEMATICS AND STATISTICS

Final Exam Key

MATH 232 Spring Semester, 1999

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[4] **1.** (a) **Define the term "reduced row-echelon matrix".**

ANSWER BOX

A matrix is a *reduced row-echelon matrix* if it has the following properties:

1. any zero row lies below all nonzero rows
2. all pivots are equal to 1
3. every other entry in the same column as a pivot is equal to 0
4. each pivot lies to the right of all pivots above it.

[3] (b) **Find a reduced row-echelon matrix row-equivalent to**

$$\begin{bmatrix} 0 & 2 & 1 & -1 \\ 1 & 1 & -2 & 1 \\ -1 & 1 & 3 & -1 \end{bmatrix}$$

ANSWER

$$\begin{bmatrix} 1 & 0 & -5/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[7] **2.** Let

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & -1 & -1 \\ 0 & 2 & 1 & 2 \\ 1 & 4 & 1 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = [1, -2, 1, 0]$$

Find the general solution of the system $Ax = \boldsymbol{b}$.

ANSWER

$$\boldsymbol{x} = \begin{bmatrix} -2 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

3. Let V denote the subspace of $\mathbb{R}^6$ spanned by

$$\begin{aligned} \boldsymbol{a}_1 &= [2,-1,3,4,1,2] \\ \boldsymbol{a}_2 &= [-2,5,3,2,1,-4] \\ \boldsymbol{a}_3 &= [2,4,6,5,2,1] \\ \boldsymbol{a}_4 &= [1,-1,1,-1,2,2] \\ \boldsymbol{a}_5 &= [1,8,10,2,5,-1] \\ \boldsymbol{a}_6 &= [3,0,0,2,1,5] \end{aligned}$$

and $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 \end{bmatrix}$ be the 6×6 matrix whose columns are $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, $\boldsymbol{a}_5$, $\boldsymbol{a}_6$.

By elementary row operations A is converted to

$$H = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[2] (a) **What is the dimension of V?**

ANSWER
4

[3] (b) **Write down a basis for V.**

ANSWER
$\{\boldsymbol{a}_1,\, \boldsymbol{a}_2,\, \boldsymbol{a}_4,\, \boldsymbol{a}_5\}$

[3] (c) **Write down a basis for the nullspace of A.**

ANSWER
$\{[-3,-1,1,0,0,0],\, [1,1,0,1,0,1]\}$

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation.

[2] (a) **Define ker(T).**

ANSWER
$\text{ker}(T) = \{\boldsymbol{a} \in \mathbb{R}^4 : T(\boldsymbol{a}) = \boldsymbol{0}\}$

[2] (b) **Define range(T).**

ANSWER
$\text{range}(T) = \{T(\boldsymbol{a}) : \boldsymbol{a} \in \mathbb{R}^4\}$

[4] (c) It is given that

$$T([1,0,0,0]) = [1,2,3], \qquad T([1,1,0,0]) = [2,3,4]$$
$$T([1,1,1,0]) = [3,4,5], \qquad T([1,1,1,1]) = [4,5,6]\,.$$

Find the standard matrix representation of T.

ANSWER

$$[T]_{\mathcal{E}} = [T(\boldsymbol{e}_1)\ T(\boldsymbol{e}_2)\ T(\boldsymbol{e}_3)\ T(\boldsymbol{e}_4)]$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

[6] **5.** On a separate sheet circulated with the exam you have the definition of a *vector space over* $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$.

From the axioms listed on the sheet, prove that, for all vectors $\boldsymbol{a}$, in V,

$$-\boldsymbol{a} = (-1)\boldsymbol{a}\,.$$

ANSWER

$$\begin{aligned}
(-1)\boldsymbol{a} &= \boldsymbol{0} + (\boldsymbol{0} + (-1)\boldsymbol{a}) && \text{by A3} \\
&= (\boldsymbol{a} + (-\boldsymbol{a})) + ((\boldsymbol{a} + (-\boldsymbol{a})) + (-1)\boldsymbol{a}) && \text{by A4} \\
&= (1\boldsymbol{a} + (-\boldsymbol{a})) + ((1\boldsymbol{a} + (-\boldsymbol{a})) + (-1)\boldsymbol{a}) && \text{by S4} \\
&= (1\boldsymbol{a} + (1\boldsymbol{a} + (-1)\boldsymbol{a})) + ((-\boldsymbol{a}) + (-\boldsymbol{a})) && \text{by A1, A2} \\
&= (1 + 1 + (-1))\boldsymbol{a} + ((-\boldsymbol{a}) + (-\boldsymbol{a})) && \text{by S2} \\
&= \boldsymbol{a} + ((-\boldsymbol{a}) + (-\boldsymbol{a})) && \text{by S4} \\
&= (\boldsymbol{a} + (-\boldsymbol{a})) + (-\boldsymbol{a}) && \text{by A1} \\
&= \boldsymbol{0} + (-\boldsymbol{a}) && \text{by A3} \\
&= -\boldsymbol{a} && \text{by A4.}
\end{aligned}$$

[4] **6.** Let ${}^{\mathbb{R}}\mathbb{R}$ denote the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ and

$$V = \left\{ f \in {}^{\mathbb{R}}\mathbb{R} : (\forall x,\, y \in \mathbb{R})\,(xy > 0 \text{ implies } f(x) = f(y)) \right\} .$$

Find a basis of V as a subspace of ${}^{\mathbb{R}}\mathbb{R}$.

ANSWER

One basis is $\{f_-,\, f_0,\, f_+\}$, where

$$f_-(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases} \qquad f_0(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \qquad f_+(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \,. \end{cases}$$

[6] **7.** Let $A \in \mathbb{R}^{n \times n}$ and ρ be an elementary row operation.

Describe how $\det \rho(A)$ depends on ρ and $\det(A)$.

ANSWER

$$\det \rho(A) = \begin{cases} \det(A) & \text{if } \rho \text{ is } R_i \to R_i + cR_j \\ c\det(A) & \text{if } \rho \text{ is } R_i \to cR_i \\ -\det(A) & \text{if } \rho \text{ is } R_i \leftrightarrow R_j \end{cases}$$

[4] **8.** **Evaluate the determinant**

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 0 & -1 & 1 & 2 \\ 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 & 3 \end{bmatrix}$$

ANSWER

1

EXPLANATION

Applying the row operations $R_5 \to R_5 + R_4$, $R_2 \leftrightarrow R_4$, $R_3 \to R_3 - R_1$, $R_5 \to R_5 - R_1$, we get the matrix

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & -4 & 1 & -3 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Note that because of the row swap, the determinant of the new matrix is minus the determinant of the original. Expanding by the first column, the determinant of the latter matrix is

$$\begin{vmatrix} -4 & 1 & -3 \\ -1 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -4 & -3 \\ 1 & 2 \end{vmatrix} = -1\,.$$

9. Let A denote the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

[4] (a) **Find the eigenvalues of A.**

ANSWER

$2,\ 3$

[4] (b) **Find the eigenspaces of A.**

ANSWER

$E_2 = \mathsf{sp}([1,2,3])$
$E_3 = \mathsf{sp}([1,1,1])$

[2] (c) **Is A diagonalizable?**

Justify your answer.

ANSWER

No, because the sum of the dimensions of the eigenspaces is less than the size of the matrix.

EXPLANATION

$$\begin{aligned}
|\lambda I - A| &= \begin{vmatrix} \lambda-3 & -1 & 1 \\ \lambda-3 & \lambda-3 & 1 \\ \lambda-3 & -1 & \lambda-2 \end{vmatrix} && C_1 \to C_1 + C_2,\ C_1 \to C_1 + C_3 \\
&= (\lambda-3)\begin{vmatrix} 1 & -1 & 1 \\ 1 & \lambda-3 & 1 \\ 1 & -1 & \lambda-2 \end{vmatrix} && C_1 \to (\lambda-3)C_1 \\
&= (\lambda-3)\begin{vmatrix} 1 & -1 & 0 \\ 1 & \lambda-3 & 0 \\ 1 & -1 & \lambda-3 \end{vmatrix} && C_3 \to C_3 - C_1 \\
&= (\lambda-3)^2(\lambda-2) && \text{expansion by column 3}
\end{aligned}$$

$$E_3 = \mathsf{nullspace}\,(3I - A) = \mathsf{nullspace}\left(\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}\right) = \mathsf{sp}([1,1,1])\,.$$

$$E_2 = \mathsf{nullspace}\,(2I - A) = \mathsf{nullspace}\left(\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & 1 & 0 \end{bmatrix}\right) = \mathsf{sp}([1,2,3])\,.$$

10. Let A denote the matrix $\begin{bmatrix} 3/10 & 4/10 & 3/10 \\ 1/10 & 2/10 & 1/10 \\ 6/10 & 4/10 & 6/10 \end{bmatrix}$.

It is given that the eigenvalues of A are 0, 1, $1/10$.

[4] (a) **Find C such that $C^{-1}AC$ is a diagonal matrix.**

ANSWER

$$C = \begin{bmatrix} 14 & 1 & 1 \\ 5 & 1 & 0 \\ 26 & -2 & -1 \end{bmatrix}$$

[4] (b) **Compute $\lim_{n\to\infty} A^n$.**

ANSWER

$$\lim_{n\to\infty} A^n = \frac{1}{45}\begin{bmatrix} 14 \\ 5 \\ 26 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

EXPLANATION

$$E_0 = \text{nullspace}(0I - A) = \text{nullspace}(10A) = \text{nullspace}\left(\begin{bmatrix} 3 & 4 & 3 \\ 1 & 2 & 1 \\ 6 & 4 & 6 \end{bmatrix}\right) = \text{sp}([1, 0, -1])$$

$$E_1 = \text{nullspace}(I - A) = \text{nullspace}(10A - 10I) = \text{nullspace}\left(\begin{bmatrix} -7 & 4 & 3 \\ 1 & -8 & 1 \\ 6 & 4 & -4 \end{bmatrix}\right) = \text{sp}([14, 5, 26])$$

$$E_{1/10} = \text{nullspace}((1/10)I - A) = \text{nullspace}(10A - I) = \text{nullspace}\left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 1 & 1 \\ 6 & 4 & 5 \end{bmatrix}\right) = \text{sp}([1, 1, -2])$$

Let C be the matrix displayed in the answer box above. Then $C^{-1}AC = D$, where D is the diagonal entry with diagonal entries 1, $1/10$, and 0 respectively. Therefore $A^n = CD^nC^{-1}$. Now

$$\lim_{n\to\infty} D^n = \lim_{n\to\infty}\begin{bmatrix} 1^n & 0 & 0 \\ 0 & 1/10^n & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$\lim_{n\to\infty} A^n = C\left(\lim_{n\to\infty} D^n\right)C^{-1} = C\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}C^{-1} = \frac{1}{45}\begin{bmatrix} 14 \\ 5 \\ 26 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Note that $[14, 5, 26]$ is the first column of C, while $(1/45)[1, 1, 1]$ is the first row of C^{-1}. Note that $\det(C) = -45$ and that we can compute the first row of C^{-1} from the formula $C^{-1} = (1/\det(C))\,\text{adj}(C)$, obtaining

$$(1/\det(C))\left[\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right] = (-1/45)[-1, -1, -1].$$

11. Let W denote the subspace of $\mathbb{R}^4$ defined by $W = \text{sp}([1,1,-1,1],[1,1,0,0])$.
Let $\boldsymbol{b} = [b_1, b_2, b_3, b_4]$ be a general vector in $\mathbb{R}^4$.

[3] (a) **Find the orthogonal complement $W^\perp$ of W.**

ANSWER

$$W^\perp = \text{sp}([1,-1,0,0],[0,0,1,1])$$

[3] (b) **Find an orthogonal basis for W.**

ANSWER

$$\{[1,1,0,0],[0,0,-1,1]\}$$

[3] (c) **Find $\boldsymbol{b}_W$ the projection of $\boldsymbol{b}$ on W. Your answer should give the components of $\boldsymbol{b}_W$ explicitly in terms of b_1, b_2, b_3, b_4.**

ANSWER

$$\frac{1}{2}[b_1+b_2, b_1+b_2, b_3-b_4, b_4-b_3]$$

EXPLANATION

(a) $W^\perp = \text{nullspace}\left(\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}\right) = \text{nullspace}\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\right)$.

(b) Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ denote $[1,1,0,0]$, $[1,1,-1,1]$ respectively. Applying the Gram-Schmidt method we get

$$\boldsymbol{v}_1' = \boldsymbol{v}_1, \quad \boldsymbol{v}_2' = \boldsymbol{v}_2 - \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_1'}{\boldsymbol{v}_1' \cdot \boldsymbol{v}_1'}\right)\boldsymbol{v}_1' = [0,0,-1,1].$$

(c) There are three ways to compute $\boldsymbol{b}_W$. Since we have already computed an orthogonal basis for W, it is most convenient to use the formula found in §6.2 of Fraleigh and Beauregard.

$$\boldsymbol{b}_W = \left(\frac{\boldsymbol{b} \cdot \boldsymbol{v}_1'}{\boldsymbol{v}_1' \cdot \boldsymbol{v}_1'}\right)\boldsymbol{v}_1' + \left(\frac{\boldsymbol{b} \cdot \boldsymbol{v}_2'}{\boldsymbol{v}_2' \cdot \boldsymbol{v}_2'}\right)\boldsymbol{v}_2'.$$

12. Let

$$B_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \quad B_3 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

and

$$\mathcal{B} = \langle B_1,\, B_2,\, B_3 \rangle \qquad \mathcal{B}' = \langle B_4,\, B_5,\, B_6 \rangle\,.$$

Let V denote the subspace of $\mathbb{R}^{2\times 2}$ of which $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases.
Let $T : V \to V$ be the linear operator defined by

$$T(X) = X \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (X \in V)\,.$$

[5] (a) **Compute the change of basis matrix $C_{\mathcal{B},\mathcal{B}'}$.**

ANSWER

$$C_{\mathcal{B},\mathcal{B}'} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

[5] (b) **Compute the matrix $[T]_{\mathcal{B}}$ which represents T with respect to $\mathcal{B}$.**

ANSWER

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

EXPLANATION

(a) $B_1 = B_1' - B_3'$ and so $(B_1)_{\mathcal{B}'} = [1, 0, -1]$

$B_2 = B_2' + B_3'$ and so $(B_1)_{\mathcal{B}'} = [0, 1, 1]$

$B_3 = B_1' - B_2' - B_3'$ and so $(B_1)_{\mathcal{B}'} = [0, 1, 1]$

(b) $T(B_1) = B_1$ and so $T(B_1)_{\mathcal{B}} = [1, 0, 0]$

$T(B_2) = B_2$ and so $T(B_2)_{\mathcal{B}} = [0, 1, 0]$

$T(B_3) = -B_1 + B_2 + 3B_3$ and so $T(B_3)_{\mathcal{B}} = [-1, 1, 3]$

[6] **13. Find a rotation of $\mathbb{R}^3$ which diagonalizes the quadratic form**

$2x^2+3y^2+2z^2+2xy+2yz.$

ANSWER

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix}$$

EXPLANATION

Let A denote the symmetric matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Note that the given quadratic form can be written in terms of A as

$$[x \quad y \quad z]\, A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To find the eigenvalues we solve

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ -1 & \lambda-3 & -1 \\ 0 & -1 & \lambda-2 \end{vmatrix} = 0\,.$$

It turns out that the eigenvalues are $\lambda = 1,\, 2,\, 4$. Next we compute

$$E_1 = \mathsf{sp}([1,-1,1]), \qquad E_2 = \mathsf{sp}([-1,0,1]), \qquad E_4 = \mathsf{sp}([1,2,1])\,.$$

Now we choose C so that its columns are eigenvectors of A which form an orthonormal basis of $\mathbb{R}^3$ and so that $|C| = 1$. The substitution $[x,y,z] = C[\overline{x},\overline{y},\overline{z}]$ converts the given quadratic form to

$$\left(C \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} \right)^T A \left(C \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} \right) = [\overline{x} \quad \overline{y} \quad \overline{z}]\, C^T A C \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix} = \overline{x}^2 + 2\overline{y}^2 + 4\overline{z}^2\,.$$

Note that $[x,y,z] = C[\overline{x},\overline{y},\overline{z}]$ is a rotation because C is orthogonal and $|C| = 1$.

[6] **14.** **Explain briefly the role of diagonalization of 2×2 matrices in classifying curves in $\mathbb{R}^2$ whose equations have the form**

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

with $a, b, c, d, e, f \in \mathbb{R}$**.**

ANSWER

Diagonalization of the quadratic form $ax^2 + 2bxy + cy^2$ by a rotation of axes, followed by a translation of axes, allows us to find a rectangular coordinate system, with the same scale, with respect to which the equation of the curve has one of the 'standard' forms:

$$a'\overline{\overline{x}}^2 + c'\overline{\overline{y}}^2 + f' = 0 \ \text{ (with } a'c' \neq 0\text{)}, \qquad \overline{\overline{x}}^2 + e'\overline{\overline{y}} = 0, \qquad \overline{\overline{x}}^2 + f' = 0\,.$$

From the standard form the nature of the curve is immediately clear. Thus in this context diagonalization should be seen as the first step in a two-step process.

Diagonalization of the quadratic form $ax^2 + 2bxy + cy^2$ is the same as diagonalization of the symmetric 2×2 matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

The mechanics of diagonalization are exactly as illustrated for a 3×3 matrix in the answer to the previous question.

The immediate purpose of the diagonalization of A is to give a rotation $[x, y, z] = C[\overline{x}, \overline{y}, \overline{z}]$ of axes which converts the given equation to a similar equation in $\overline{x}$, $\overline{y}$ in which the coefficient of $\overline{x}\,\overline{y}$ is 0.