MATH 232
Final Exam
December 15, 1999

ANSWER KEY

Questions 1, 4, 5, 7, 8b, 10 and 12 are identical in both versions of the exam.
For the remaining questions, the text of the question from the green booklet is printed, then the solution for the green booklet is given and then a synopsis of the solution for the pink booklet follows immediately.

$\boxed{1.}$

Question:
(a) Give a precise description of the kinds of row operation which are permitted in bringing a matrix to reduced row-echelon form.
(b) Define an elementary matrix.

Marking:
(a) - 4 marks
(b) - 2 marks

6 marks total

Solution:
(a) These operations are called the *elementary row operations.* They are:
- Interchange two rows in the matrix.
- Multiply a row in the matrix by a non-zero constant.
- Replace a row in the matrix with the sum of itself and a multiple of a different row in the matrix.
(See page 56.)

(b) Any matrix that can be obtained from an identity matrix by means of one elementary row operation is called an *elementary matrix.*
(Definition 1.14, page 65)

$\boxed{2.}$

Question:
Find a basis for the set of solutions to the system

$$\begin{array}{rcrcrcrcl} 2x_1 & - & x_2 & - & 6x_3 & + & 10x_4 & = & 0 \\ -x_1 & + & 3x_2 & + & 8x_3 & - & 15x_4 & = & 0\,. \end{array}$$

Marking:
6 marks

Solution: (See Section 1.6 in the book.)

green papers

This is a homogeneous linear system and therefore we do not need to work with the augmented matrix. By Gauss reduction we find the reduced row-echelon form of the coefficient matrix:

$$\begin{bmatrix} 2 & -1 & -6 & 10 \\ -1 & 3 & 8 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

Hence the general solution to the given system is

$$\begin{bmatrix} 2x_3 - 3x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

and a basis for the space of solutions is

$$\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

pink papers

The Gauss reduction of the coefficient matrix is

$$\begin{bmatrix} 2 & -1 & 3 & -10 \\ -1 & 3 & 1 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

and a basis for the space of solutions is

$$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

3.

Question:
Let $V = \text{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5)$ denote the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned} \boldsymbol{a}_1 &= [2, 2, 1, -1, 0] \\ \boldsymbol{a}_2 &= [-1, 1, 1, 2, 2] \\ \boldsymbol{a}_3 &= [7, 1, -1, -8, -6] \\ \boldsymbol{a}_4 &= [0, 8, 6, 6, 8] \\ \boldsymbol{a}_5 &= [1, 1, 1, -1, -3] \end{aligned}$$

and $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{bmatrix}$ be the 5×5 matrix whose columns are $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, $\boldsymbol{a}_5$.

By elementary row operations A is converted to

$$H = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Write down a basis for V.
(b) Write down a basis for the row space of A.
(c) Determine the rank of A. Give a reason for your answer.

Marking:
(a) - 2 marks
(b) - 2 marks
(c) - 2 marks

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6 marks total

Solution:
We notice that H is the reduced row-echelon form of A, and that V is the column space of A.

green papers

(a) For a basis for V we use the columns of A corresponding to the columns of H containing pivots. Thus

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_5\} = \{[2, 2, 1, -1, 0],\ [-1, 1, 1, 2, 2],\ [1, 1, 1, -1, -3]\}$$

is a basis for V. (See page 138.)
(b) For a basis of the row space of A we use the nonzero rows of H:

$$\{[1, 0, 2, 2, 0],\ [0, 1, -3, 4, 0],\ [0, 0, 0, 0, 1]\}$$

(c) By Theorem 2.4 (page 137) the rank of A is the dimension of the row space of A, which at the same time is the dimension of the column space of A. Using the results of parts (a) and (b) we get rank$(A) = 3$.

pink papers

(a) For a basis for V we use the columns of A corresponding to the columns of H containing pivots. Thus

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_5\} = \{[2, 3, 1, -1, 0],\ [-1, 1, 1, 2, 2],\ [1, -1, 1, -1, -3]\}$$

is a basis for V. (See page 138.)
(b) For a basis of the row space of A we use the nonzero rows of H:

$$\{[1, 0, 2, 2, 0],\ [0, 1, -3, 4, 0],\ [0, 0, 0, 0, 1]\}$$

(c) By Theorem 2.4 (page 137) the rank of A is the dimension of the row space of A, which at the same time is the dimension of the column space of A. Using the results of parts (a) and (b) we get rank$(A) = 3$.

$\boxed{\mathbf{4.}}$

Question:
On a separate sheet circulated with this exam you find the definition of a vector space over $\mathbb{R}$.
Let V be a vector space over $\mathbb{R}$. From the axioms listed in the definition, prove that, for any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in V there exists a unique vector $\boldsymbol{x}$ in V such that $\boldsymbol{v} + \boldsymbol{x} = \boldsymbol{w}$.

Marking:
7 marks

Solution: (See Section 3.1 in the book.)
Assume $\boldsymbol{v} + \boldsymbol{x} = \boldsymbol{w}$. By the definition of the vector space we get

$$
\begin{array}{rcll}
\boldsymbol{v} + \boldsymbol{x} & = & \boldsymbol{w} & \\
-\boldsymbol{v} + (\boldsymbol{v} + \boldsymbol{x}) & = & -\boldsymbol{v} + \boldsymbol{w} & \\
(-\boldsymbol{v} + \boldsymbol{v}) + \boldsymbol{x} & = & -\boldsymbol{v} + \boldsymbol{w} & \text{(by A1)} \\
(\boldsymbol{v} + (-\boldsymbol{v})) + \boldsymbol{x} & = & -\boldsymbol{v} + \boldsymbol{w} & \text{(by A2)} \\
\mathbf{0} + \boldsymbol{x} & = & -\boldsymbol{v} + \boldsymbol{w} & \text{(by A4)} \\
\boldsymbol{x} & = & -\boldsymbol{v} + \boldsymbol{w} & \text{(by A3).}
\end{array}
$$

Because the vector addition and the additive inverse are (uniquely defined) functions, the vector $-\boldsymbol{v} + \boldsymbol{w}$ is unique. This completes the proof.

$\boxed{\mathbf{5.}}$

Question:
Let V be a vector space over $\mathbb{R}$. Let W_1 and W_2 be two subspaces of V. Prove that their intersection $W_1 \cap W_2$ is a subspace of V.

Marking:
5 marks

Solution:
Using Theorem 3.2 on page 193 it is sufficient to prove that $W_1 \cap W_2$ is:
(i) nonempty,
(ii) closed under vector addition, and
(iii) closed under scalar multiplication.

(i) Let $\mathbf{0}$ be the zero vector of V. Since W_1 and W_2 are subspaces of V, we have $\mathbf{0} \in W_1$, $\mathbf{0} \in W_2$ and so $\mathbf{0} \in W_1 \cap W_2$. Thus $W_1 \cap W_2$ is nonempty.

(ii) Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be arbitrary vectors in $W_1 \cap W_2$. Then $\boldsymbol{v}_1 \in W_1$, $\boldsymbol{v}_2 \in W_1$ and $\boldsymbol{v}_1 \in W_2$, $\boldsymbol{v}_2 \in W_2$. Since W_1 and W_2 are subspaces of V, we have $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in W_1$ and $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in W_2$. Thus $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in W_1 \cap W_2$ and $W_1 \cap W_2$ is closed under vector addition.
(iii) Let $c \in \mathbb{R}$ be an arbitrary scalar and let $\boldsymbol{v}$ be an arbitrary vector in $W_1 \cap W_2$. Then $\boldsymbol{v} \in W_1$ and $\boldsymbol{v} \in W_2$. Since W_1 and W_2 are subspaces of V, we have $c\boldsymbol{v} \in W_1$ and $c\boldsymbol{v} \in W_2$. Thus $c\boldsymbol{v} \in W_1 \cap W_2$ and $W_1 \cap W_2$ is closed under scalar multiplication.

6.

Question:
Let $\mathbb{R}^{2\times 2}$ denote the vector space of all 2×2 real matrices, using as vector addition and scalar multiplication the usual addition of matrices and multiplication of a matrix by a scalar.
Given are four matrices

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}.$$

It is given that $\mathcal{B} = (\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4)$ is an ordered basis for $\mathbb{R}^{2\times 2}$.
Let

$$\boldsymbol{v} = \begin{bmatrix} 2 & -1 \\ 6 & 6 \end{bmatrix}.$$

Find the coordinate vector $\boldsymbol{v}_{\mathcal{B}}$ of $\boldsymbol{v}$ relative to $\mathcal{B}$.

Marking:
6 marks

Solution: (See Section 3.3 and Example 5 on pages 210–211.)
Let $\mathcal{C} = (\boldsymbol{w}_1,\ \boldsymbol{w}_2,\ \boldsymbol{w}_3,\ \boldsymbol{w}_4)$ be another ordered basis for $\mathbb{R}^{2\times 2}$, where

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{w}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{w}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{w}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The space $\mathbb{R}^{2\times 2}$ is isomorphic to $\mathbb{R}^4$ via the coordinatization isomorphism $\boldsymbol{x} \mapsto \boldsymbol{x}_{\mathcal{C}}$ ($\boldsymbol{x} \in \mathbb{R}^{2\times 2}$, $\boldsymbol{x}_{C} \in \mathbb{R}^4$). Once this isomorphism is introduced, we can find the desired coordinate vector $\boldsymbol{v}_{\mathcal{B}}$ using standard techniques for $\mathbb{R}^n$.

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Using the method on page 207 we find

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 2 & 2 & -1 & -3 & -1 \\ -1 & -1 & 3 & 2 & 6 \\ 2 & 4 & 1 & 0 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

and so $\boldsymbol{v}_{\mathcal{B}} = [2, 0, 2, 1]$.

Using the method on page 207 we find

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 2 \\ 2 & 2 & -1 & -3 & -1 \\ -1 & -1 & 3 & 2 & 8 \\ 2 & 4 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

and so $\boldsymbol{v}_{\mathcal{B}} = [2, -1, 3, 0]$.

7.

Question:

Let F be the vector space of all functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let W be the subspace of F spanned by the four functions 1, x, e^x and xe^x. It is given that $\mathcal{B} = (1,\, x,\, e^x,\, xe^x)$ is an ordered basis for W. Given are two linear transformations $T_1 : W \to W$ and $T_2 : W \to W$ defined by

$T_1(f) = f'$ (the derivative of f with respect to x) for all $f \in W$

$T_2(f) = f''$ (the second derivative of f with respect to x) for all $f \in W$.

Let A_1 be the matrix representation of T_1 relative to $\mathcal{B}, \mathcal{B}$ and let A_2 be the matrix representation of T_2 relative to $\mathcal{B}, \mathcal{B}$.

(a) Find the matrix A_1.

(b) Decide whether the transformation T_1 is invertible. Justify your answer.

(c) Use the composition of linear transformations to discover a simple relation between A_2 and A_1. Justify your answer. Do not compute A_2 explicitly, just express it in terms of A_1.

Marking:

(a) - 5 marks

(b) - 3 marks

(c) - 2 marks

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10 marks total

Solution:

(a) We have $T_1(1) = 0$, $T_1(x) = 1$, $T_1(e^x) = e^x$ and $T_1(xe^x) = e^x + xe^x$.
Using Theorem 3.10 and Definition 3.11 on page 223 we find

$$A_1 = [T_1(1)_{\mathcal{B}} \;\; T_1(x)_{\mathcal{B}} \;\; T_1(e^x)_{\mathcal{B}} \;\; T_1(xe^x)_{\mathcal{B}}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) The transformation T_1 is not invertible. This can be seen in many different ways, for example:
- Applying Theorem 3.8 on page 220 and noticing that $T_1(0) = T_1(1) = 0$, so T_1 is not one-to-one, see Equation (6) on page 220. Or: (see the next page ...)

- If the inverse linear transformation T_1^{-1} exists, then the matrix representation of T_1^{-1} relative to $\mathcal{B}, \mathcal{B}$ is $(A_1)^{-1}$. But the matrix A_1 is not invertible (it contains a column of zeros).

(c) The key observation is $f'' = (f')'$, that is,

$$T_2 = T_1 \circ T_1 \qquad (*)$$

where $\circ$ denotes the composition of linear transformations, see top of page 214. After recalling that A_1 is the matrix representation for T_1 and A_2 is the matrix representation for T_2 we have for any $\boldsymbol{w} \in W$

$$A_2 \boldsymbol{w}_{\mathcal{B}} = A_1(A_1 \boldsymbol{w}_{\mathcal{B}}) = (A_1^2)\boldsymbol{w}_{\mathcal{B}}$$

and, since the matrix representation relative to a fixed pair of bases is unique, we have the desired relation between A_2 and A_1:

$$A_2 = A_1^2$$

This relation also follows at once from (*) if one remembers that the composition of linear transformations corresponds to the multiplication of their matrix representations, see for example the box on page 150, or the first equation on page 397, and many other places throughout the book.

$\boxed{8.}$

Question:
(a) Given are three points $P = (3, -1)$, $Q = (2, 2)$ and $R = (-1, 7)$. Find the area of the triangle PQR.
(b) State the row-interchange property for determinants of square matrices. Use it to prove: If two rows of a square matrix A are equal, then $\det(A) = 0$.

Marking:
(a) - 4 marks
(b) - 4 marks

8 marks total

Solution: first (b), then (a)

(b) The row-interchange property for determinants of square matrices states the following: If two different rows of a square matrix A are interchanged, the determinant of the resulting matrix is $-\det(A)$. (See Property 2 on page 256.)
Let R_i denote the i-th row of A, and assume that R_j and R_k are equal ($j \neq k$). Let A' be the matrix obtained from A by interchanging R_j and R_k. Then $A' = A$, and $\det(A') = -\det(A)$ by the above property. Thus $\det(A) = -\det(A)$ which is only possible if $\det(A) = 0$. (See Property 3 on page 257.)

$\boxed{\text{green papers}}$

(a) Define the vectors

$$\begin{aligned} \boldsymbol{a} &= [2,2]-[3,-1]=[-1,3] \\ \boldsymbol{b} &= [-1,7]-[3,-1]=[-4,8]\,. \end{aligned}$$

The area of the parallelogram determined by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the absolute value of the determinant $\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{vmatrix}$, that is, the absolute value of

$$\begin{vmatrix} -1 & 3 \\ -4 & 8 \end{vmatrix} = (-1)\cdot 8 - 3\cdot(-4) = 4\,.$$

The area of the triangle PQR is one half of the area of the parallelogram, that is $4/2=2$. (See page 239.)

$\boxed{\text{pink papers}}$

(a) Define the vectors

$$\begin{aligned} \boldsymbol{a} &= [2,2]-[3,-1]=[-1,3] \\ \boldsymbol{b} &= [-1,5]-[3,-1]=[-4,6]\,. \end{aligned}$$

The area of the parallelogram determined by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the absolute value of the determinant $\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{vmatrix}$, that is, the absolute value of

$$\begin{vmatrix} -1 & 3 \\ -4 & 6 \end{vmatrix} = (-1)\cdot 6 - 3\cdot(-4) = 6\,.$$

The area of the triangle PQR is one half of the area of the parallelogram, that is $6/2=3$. (See page 239.)

$\boxed{9.}$

Question:
Evaluate the determinant.

Marking:
4 marks

Solution: See the next page. (See Theorem 4.2 on page 254 and Property 5 on page 258.)

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$$\begin{vmatrix} 0 & 3 & 3 & 5 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 3 & -4 \\ -2 & 0 & 1 & 7 \end{vmatrix} = 3 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -4 \\ -2 & 1 & 7 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -4 \\ 0 & -3 & 9 \end{vmatrix}$$

$$= -3 \cdot \begin{vmatrix} 3 & -4 \\ -3 & 9 \end{vmatrix} = -3 \cdot (3 \cdot 9 - (-4) \cdot (-3)) = -3 \cdot 15 = -45.$$

pink papers

$$\begin{vmatrix} 0 & 2 & 3 & 5 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 3 & -4 \\ -3 & 0 & 1 & 7 \end{vmatrix} = 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -4 \\ -3 & 1 & 7 \end{vmatrix} = -2 \cdot \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -4 \\ 0 & -5 & 10 \end{vmatrix}$$

$$= -2 \cdot \begin{vmatrix} 3 & -4 \\ -5 & 10 \end{vmatrix} = -2 \cdot (3 \cdot 10 - (-4) \cdot (-5)) = -2 \cdot 10 = -20.$$

10.

Question:

Let

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding eigenspaces of A.
(b) Use diagonalization to compute A^{2000}. Give your answer in the form of a single 3×3 matrix.
(c) Decide whether the matrix

$$B = \begin{bmatrix} 4 & 3 \\ 0 & 4 \end{bmatrix}$$

is diagonalizable. Justify your answer.

Marking:

(a) - 5 marks
(b) - 3 marks
(c) - 3 marks

11 marks total

Solution: (See Sections 5.1 and 5.2.)
(a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & -2 & 1 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)((1-\lambda)(-2-\lambda)-(-1)2) = (1-\lambda)(\lambda+\lambda^2) = (1-\lambda)\lambda(1+\lambda).\end{aligned}$$

Thus the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = -1$.

Since all eigenvalues are different, the corresponding eigenspaces E_{λ_i} $(i = 1, 2, 3)$ all have dimension 1. The bases for the eigenspaces E_{λ_i} are found by solving the homogeneous linear systems $(A - \lambda_i I)\boldsymbol{x} = \mathbf{0}$. By Gauss reduction we obtain

$$\begin{bmatrix} 1-1 & -2 & 1 \\ 0 & 1-1 & -1 \\ 0 & 2 & -2-1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-0 & -2 & 1 \\ 0 & 1-0 & -1 \\ 0 & 2 & -2-0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-(-1) & -2 & 1 \\ 0 & 1-(-1) & -1 \\ 0 & 2 & -2-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the eigenspaces are $E_1 = \mathrm{sp}([1,0,0])$, $E_0 = \mathrm{sp}([1,1,1])$ and $E_{-1} = \mathrm{sp}([0,1/2,1]) = \mathrm{sp}([0,1,2])$.

(b) By Theorem 5.3 on page 308, A is diagonalizable. Let

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then $C^{-1}AC = D$ and $A^k = CD^kC^{-1}$. (See page 307.) The inverse of C can be computed, for example, using the method on page 80. We obtain

$$C^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and so

$$\begin{aligned}A^{2000} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2000}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}.\end{aligned}$$

$\boxed{\text{green papers}}$

(c) Let

$$B = \begin{bmatrix} 4 & 3 \\ 0 & 4 \end{bmatrix}.$$

The characteristic polynomial of B is $\det(B - \lambda I) = (4-\lambda)^2 - 3 \cdot 0 = (4-\lambda)^2$ and so $\lambda_1 = 4$ is an eigenvalue of B with algebraic multiplicity 2. By solving the homogeneous system $(B - 4I)\boldsymbol{x} = \mathbf{0}$ we find that the eigenspace corresponding to the eigenvalue 4 is $E_4 = \mathrm{sp}([1, 0])$. Thus the geometric multiplicity of the eigenvalue 4 is 1. By Theorem 5.4 (page 313), the matrix B is not diagonalizable.

$\boxed{\text{pink papers}}$

(c) Let

$$B = \begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix}.$$

The characteristic polynomial of B is $\det(B - \lambda I) = (3-\lambda)^2 - (-2) \cdot 0 = (3-\lambda)^2$ and so $\lambda_1 = 3$ is an eigenvalue of B with algebraic multiplicity 2. By solving the homogeneous system $(B - 3I)\boldsymbol{x} = \mathbf{0}$ we find that the eigenspace corresponding to the eigenvalue 3 is $E_3 = \mathrm{sp}([1, 0])$. Thus the geometric multiplicity of the eigenvalue 3 is 1. By Theorem 5.4 (page 313), the matrix B is not diagonalizable.

$\boxed{\textbf{11.}}$

Question:

(a) Let $\boldsymbol{a} = [2, 1, -1]$ and $\boldsymbol{b} = [-1, 3, 0]$. Find the projection of $\boldsymbol{b}$ on $\mathrm{sp}(\boldsymbol{a})$.
(b) Let W be the subspace of $\mathbb{R}^3$ defined by

$$W = \{[x, y, z] \in \mathbb{R}^3 \mid x + y - z = 0\}.$$

Write down the basis for $W^\perp$, the orthogonal complement of W.
(c) Let $\boldsymbol{c} = [2, 1, 6]$. Find the projection of $\boldsymbol{c}$ on W.

Marking:

(a) - 2 marks
(b) - 3 marks
(c) - 4 marks

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9 marks total

Solution:

$\boxed{\text{green papers}}$

(a) Let $\boldsymbol{p}$ be the projection of $\boldsymbol{b}$ on $\mathrm{sp}(\boldsymbol{a})$. Using Equation (1) on page 327 we find

$$\boldsymbol{p} = \frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}} \boldsymbol{a} = \frac{-1 \cdot 2 + 3 \cdot 1 + 0 \cdot (-1)}{2 \cdot 2 + 1 \cdot 1 + (-1) \cdot (-1)} [2, 1, -1] = \frac{1}{6} [2, 1, -1].$$

(b) This is a special instance of Illustration 3 on page 331, which in this case takes the form

$$W = \{[x, y, z] \in \mathbb{R}^3 \mid [1, 1, -1] \cdot [x, y, z] = 0\}$$

and so

$$W^\perp = \mathsf{sp}([1, 1, -1])\,.$$

An alternative solution is to start by finding a basis for W. One can view the defining equation of W, that is $x + y - z = 0$, as a homogeneous linear system whose coefficient matrix $[1\ 1\ \ -1]$ already is in the reduced row-echelon form. By the standard methods of Section 1.6 we find that $B = \{[-1, 1, 0], [1, 0, 1]\}$ is a basis for W and then using the method on page 330 (finding the nullspace of the matrix that has the two vectors in B as its rows) we arrive at the same result $W^\perp = \mathsf{sp}([1, 1, -1])$.

(c) Let $\boldsymbol{c}_W$ denote the projection of $\boldsymbol{c}$ on W, thus $\boldsymbol{c}_W \in W$. Recall that $\boldsymbol{c}$ can be written uniquely as

$$\boldsymbol{c} = \boldsymbol{c}_W + \boldsymbol{c}_{W^\perp} \qquad\qquad (*)$$

where $\boldsymbol{c}_{W^\perp} \in W^\perp$, in fact $\boldsymbol{c}_{W^\perp}$ is the projection of $\boldsymbol{c}$ on $W^\perp$. Since $W^\perp = \mathsf{sp}([1, 1, -1])$, see the result of part (b), it is easy to compute

$$\boldsymbol{c}_{W^\perp} = \frac{\boldsymbol{c} \cdot [1, 1, -1]}{[1, 1, -1] \cdot [1, 1, -1]}[1, 1, -1] = \frac{-3}{3}[1, 1, -1] = [-1, -1, 1]$$

and from (*) we get

$$\boldsymbol{c}_W = \boldsymbol{c} - \boldsymbol{c}_{W^\perp} = [2, 1, 6] - [-1, -1, 1] = [3, 2, 5]\,.$$

This approach is taken in Example 5 on page 334.

Alternatively, one can use the standard method for finding projections of vectors on subspaces, see the box on page 333. Utilizing the basis B that we found in part (b) we take $B' = \{[-1, 1, 0], [1, 0, 1], [1, 1, -1]\}$ as a basis for $\mathbb{R}^3$ (with the first two vectors spanning W and the third vector spanning $W^\perp$) and then we find (using the method on page 207) that the coordinate vector of $\boldsymbol{c}$ relative to B' is

$$\boldsymbol{c}_{B'} = [2, 5, -1]\,.$$

Then we get

$$\boldsymbol{c}_W = 2[-1, 1, 0] + 5[1, 0, 1] = [3, 2, 5]\,.$$

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The solution is completely analogous. The results for parts (a), (b) and (c) are

$$\begin{aligned} \boldsymbol{p} &= \frac{5}{6}[1, 2, -1] \\ W^\perp &= \mathsf{sp}([1, -1, 1]) \\ \boldsymbol{c}_W &= [0, 3, 3]\,. \end{aligned}$$

12.

Question:
Let $V = \text{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $\boldsymbol{a}_1 = [1,0,0,1]$, $\boldsymbol{a}_2 = [1,1,0,1]$ and $\boldsymbol{a}_3 = [0,1,-1,0]$.
(a) Find an orthogonal basis for V.
(b) Use your answer to part (a) to find an orthonormal basis for V.

Marking:
(a) - 6 marks
(b) - 2 marks

8 marks total

Solution:

(a) We use the Gram-Schmidt orthogonalization process. See Section 6.2. It is possible to take advantage of the fact that the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ are orthogonal; in this approach we set

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{a}_1 = [1,0,0,1] \\ \boldsymbol{v}_2 &= \boldsymbol{a}_3 = [0,1,-1,0] \end{aligned}$$

and

$$\boldsymbol{v}_3 = \boldsymbol{a}_2 - \frac{\boldsymbol{a}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{a}_2 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 = [1,1,0,1] - \frac{2}{2}[1,0,0,1] - \frac{1}{2}[0,1,-1,0] = [0, \frac{1}{2}, \frac{1}{2}, 0]\,.$$

Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthogonal basis for V.

On the other hand, if we use the ordering $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, then the Gram-Schmidt orthogonalization process yields the following orthogonal basis for V:

$$\{[1,0,0,1], [0,1,0,0], [0,0,-1,0]\}\,.$$

(b) To obtain an orthonormal basis for V, we take an orthogonal basis for V, obtained in part (a), and normalize each basis vector $\boldsymbol{v}_i$ by scaling it with the factor $\frac{1}{\|\boldsymbol{v}_i\|}$. For example, the second orthogonal basis from part (a) leads to the following orthonormal basis for V:

$$\left\{ \frac{1}{\sqrt{2}}[1,0,0,1], [0,1,0,0], [0,0,-1,0] \right\}$$

$\boxed{\textbf{13.}}$

Question:
The following data points are given:

$$(-2,0),\ (-1,1),\ (0,3),\ (1,6)\,.$$

Find the least-squares linear fit for these data points.

Marking:
7 marks

Solution:

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See Section 6.5 (pages 372–374). We are faced with the system of linear approximations

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} \approx \begin{bmatrix} 0 \\ 1 \\ 3 \\ 6 \end{bmatrix}.$$

The least-squares solution of this system is obtained by solving the system of two linear equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 6 \end{bmatrix}.$$

(See the box on page 374.) After performing the matrix multiplications we end up with solving the system

$$\begin{array}{rcrcr} 4r_0 & - & 2r_1 & = & 10 \\ -2r_0 & + & 6r_1 & = & 5 \end{array}$$

whose unique solution is $r_0 = 7/2$, $r_1 = 2$. Thus the least-squares linear fit for the given data points is

$$y = 2x + \frac{7}{2}\,.$$

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The solution is completely analogous. The linear system for r_0, r_1 is

$$\begin{array}{rcrcr} 4r_0 & + & 2r_1 & = & 10 \\ 2r_0 & + & 6r_1 & = & 15 \end{array}$$

and the least-squares linear fit for the given data points is

$$y = 2x + \frac{3}{2}\,.$$

$\boxed{14.}$

Question:

Let $E = (\boldsymbol{e}_1, \boldsymbol{e}_2)$ be the standard ordered basis for $\mathbb{R}^2$. Let $\boldsymbol{b}_1 = [2, 1]$, $\boldsymbol{b}_2 = [-3, -2]$ and let $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ be an ordered basis for $\mathbb{R}^2$.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T([x_1, x_2]) = [x_1 + x_2,\ x_1 - x_2]$ for every $[x_1, x_2] \in \mathbb{R}^2$.
(a) Write down the standard matrix representation of T.
(b) Find the change-of-coordinates matrix from E to B.
(c) Find the matrix representation of T relative to B.

Marking:

(a) - 2 marks
(b) - 2 marks
(c) - 3 marks

7 marks total

Solution: (See Sections 7.1 and 7.2.)

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(a) The standard matrix representation of T is the matrix representation of T relative to E. See page 146 and Example 2 on page 399. We have, *using the notation of Section 7.2,*

$$R_E = [T(\boldsymbol{e}_1)\ T(\boldsymbol{e}_2)] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(b) Let $M_B = [\boldsymbol{b}_1\ \boldsymbol{b}_2]$, see Equation 2 on page 389. By the standard method on page 391 we compute

$$\left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

and so the change-of-coordinates matrix from E to B is

$$C_{E,B} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

Comments: Of course $C_{E,B} = M_B^{-1}$ for any basis B. For 2×2 matrices computing the inverse is faster using the adjoint matrix than by Gauss reduction, see page 270. The fact that $M_B^{-1} = M_B$ in this particular example is a mere coincidence; it does not hold in general of course.

(c) See Theorem 7.1 on page 399. We have

$$R_B = C_{E,B} R_E C_{B,E}$$

and using the results of (a) and (b), along with the simple fact that $C_{B,E} = M_B$, gives

$$R_B = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 1 & -3 \end{bmatrix}.$$

$\boxed{\textbf{pink papers}}$

The solution is analogous. The results for parts (a), (b) and (c) are

$$\begin{aligned} R_E &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ C_{E,B} &= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \\ R_B &= \begin{bmatrix} -7 & 13 \\ -5 & 9 \end{bmatrix}. \end{aligned}$$