DEPARTMENT OF MATHEMATICS AND STATISTICS

First Midterm Key

MATH 232 February 3, 1999

11:30 – 12:20 a.m.

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[3] **1. Compute a vector $\boldsymbol{x} \in \mathbb{R}^3$ such that**

$$[1, -2, 1] - 2\boldsymbol{x} = [5, 2, -3].$$

ANSWER

$[-2, -2, 2]$

EXPLANATION

$$2\boldsymbol{x} = [1, -2, 1] - [5, 2, -3] = [-4, -4, 4].$$

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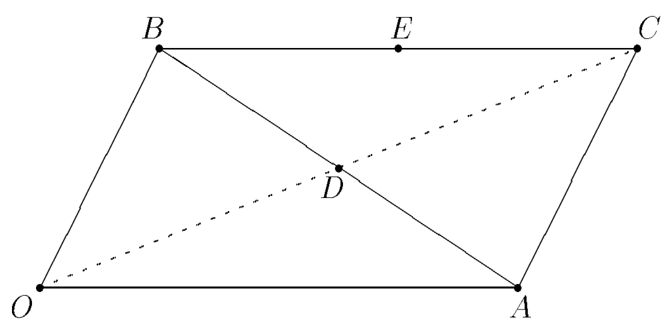
2. The diagram below shows points O, A, B, C, D, E in $\mathbb{R}^2$, O being the origin. The figure $OACB$ is a parallelogram, D is the midpoint of the line segment AB, and E is the midpoint of BC.

The points A, B have coordinate vectors $\boldsymbol{a}$, $\boldsymbol{b}$, respectively, while O has coordinate vector $\mathbf{0} = [0, 0]$.



[2] (a) **Write down the coordinate vector of D in terms of $\boldsymbol{a}$, $\boldsymbol{b}$.**

ANSWER

$\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

[2] (b) **Write down the coordinate vector of E in terms of $\boldsymbol{a}$, $\boldsymbol{b}$.**

ANSWER

$\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

EXPLANATION

Since D is midpoint of the line segment AB, its coordinate vector is the mean of the coordinate vectors of $\boldsymbol{a}$ and $\boldsymbol{b}$. C has vector $\boldsymbol{a} + \boldsymbol{b}$. Since E is midpoint of the line segment BC, its coordinate vector is the mean of $\boldsymbol{b}$ and $\boldsymbol{a} + \boldsymbol{b}$.

[2] **3.** (a) **Compute the dot product**

$$[4, 1, -1, 2] \cdot [-1, 3, -3, -1].$$

ANSWER

0

[2] (b) **Compute the norm**

$$\|[-1, 7, 5, 6, -1, 3]\|.$$

ANSWER

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[2] (c) **Find a unit vector orthogonal to**

$$[1, -2, 1].$$

ANSWER

$\left[1/\sqrt{2}, 0, -1/\sqrt{2}\right]$

EXPLANATION

Parts (a) and (b) are immediate from the definitions.

For part (c), any unit vector $\boldsymbol{x}$ such that $[x_1, x_2, x_3] \cdot [1, -2, 1] = 0$ will do.

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[2] **4.** (a) **Write down the augmented matrix of the system:**

$$\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 2 \\ 4x_1 + 2x_2 + 2x_4 = 4 \end{cases}$$

ANSWER

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 2 \\ 4 & 2 & 0 & 2 & 4 \end{array}\right]$$

[2] (b) **Convert the matrix from (a) to reduced row-echelon form by row operations.**

Write the final answer in the answer box.

ANSWER

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

EXPLANATION

One possible sequence of row operations is

$$R_2 \to (-1/2)R_2,\ R_2 \to R_2 + R_1,\ R_1 \to R_1 - R_2\,.$$

5. Consider the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4, x_5]$, $\boldsymbol{b} \in \mathbb{R}^5$, and $A \in \mathbb{R}^{4\times 5}$.

It is given that the reduced row-echelon form of the augmented matrix $[A|\boldsymbol{b}]$ is the matrix.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

[5] (a) **Find the general solution of the system $A\boldsymbol{x} = \boldsymbol{b}$ writing your final answer in vectorial form in the box below.**

ANSWER

$$\boldsymbol{x} = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad (c,\, d \in \mathbb{R})$$

OR

$$\boldsymbol{x} = [-3, 0, 1, 0, 2] + c[-2, 1, 0, 0] + d[1, 0, -2, 1, 0] \qquad (c,\, d \in \mathbb{R})$$

[2] (b) **Using your answer to (a), write down the general solution of the homogeneous system $A\boldsymbol{x} = 0$ writing your final answer in vectorial form in the box below.**

ANSWER

$$\boldsymbol{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad (c,\, d \in \mathbb{R})$$

OR

$$\boldsymbol{x} = c[-2, 1, 0, 0] + d[1, 0, -2, 1, 0] \qquad (c,\, d \in \mathbb{R})$$

EXPLANATION

Here we have used the standard algorithm for solving systems of linear equations. ■

6. Let A denote the matrix:

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

[3] (a) **Find row operations ρ_1, ρ_2, $\ldots$, ρ_k such that $\rho_k(\ldots\rho_2(\rho_1(A))\ldots) = I$.**

This can be done with three row operations but one might use more.

ANSWER

$\rho_1 = \; R_1 \to R_1 - R_3$

$\rho_2 = \; R_2 \to R_2 + 2R_1$

$\rho_3 = \; R_3 \to R_3 - R_1$

$\rho_4 =$

$\rho_5 =$

$\rho_6 =$

[2] (b) **Based on your answer to (a), express A as a product of elementary matrices.**

ANSWER

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

[2] (c) **Based on your answer to (b), express A^{-1} as a product of elementary matrices.**

ANSWER

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[3] **7.** Let vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k$ in $\mathbb{R}^n$ be given.

Describe a procedure for finding a basis of $\mathrm{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k)$.

ANSWER

Form the $n \times k$ matrix A whose columns are $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k$ in that order.
Let H be the row-reduced echelon matrix obtained from A by row-operations.
Let the pivots of H fall in the columns numbered $j_1, j_2, \ldots, j_r$. Then

$$\{\boldsymbol{a}_{j_1}, \boldsymbol{a}_{j_2}, \ldots, \boldsymbol{a}_{j_r}\}$$

is a basis for $\mathrm{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k)$. ∎

[2] **8.** Let A be the 4×5 matrix from question 5 on page 3.

Write down a basis for the nullspace of A which is defined to be

$$\left\{\boldsymbol{x} \in \mathbb{R}^5 \ : \ A\boldsymbol{x} = \boldsymbol{0}\right\}$$

ANSWER

$$\{[-2, 1, 0, 0, 0], [1, 0, -2, 1, 0]\} \ .$$ ∎

[2] **9.** The definition of *basis* says that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ is a basis for $\mathbb{R}^n$ if $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k \in \mathbb{R}^n$ are distinct and for every $\boldsymbol{v} \in \mathbb{R}^n$ there are unique $c_1, c_2, \ldots, c_k$ in $\mathbb{R}$ such that

$$c_1\boldsymbol{a}_1 + \ldots + c_k\boldsymbol{a}_k = \boldsymbol{v}\,.$$

Show from first principles that $\mathbb{R}^5$ has a basis of size 5.

ANSWER

From the definition of addition and scalar multiplication for $\mathbb{R}^5$ we have

$$[c_1, c_2, \ldots, c_5] = c_1\boldsymbol{e}_1 + \ldots + c_5\boldsymbol{e}_5$$

for all $c_1, c_2, \ldots, c_5 \in \mathbb{R}$.

This equation shows that every $\boldsymbol{c}$ in $\mathbb{R}^5$ is expressible as a linear combination of $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_5$ and that the coefficients are unique. ∎

[2] **10.** In this question you may assume the conclusion of question 9 and that *every* basis of $\mathbb{R}^5$ has size 5. You may **not** assume anything else.

Let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4$ be distinct linearly independent vectors in $\mathbb{R}^5$.

Prove that there exists a vector $\boldsymbol{c}$ in $\mathbb{R}^5$ such that $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4, \boldsymbol{c}\}$ is linearly independent.

ANSWER

If each of $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_5$ is in the span of $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4\}$, then $\text{sp}(B) = \mathbb{R}^5$. Moreover, the linear independence of B implies that, when we write $\boldsymbol{c} \in \mathbb{R}^5$ as a linear combination of $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4$, the coefficients are unique. So B is a basis for $\mathbb{R}^5$, contradiction. Thus there exists j, $1 \leq j \leq 5$, such that $\boldsymbol{e}_j \notin \text{sp}(B)$.

We claim that $B \cup \{\boldsymbol{e}_j\}$ is linearly independent. Suppose not. Then there are real numbers r_1, r_2, r_3, r_4, s not all 0 such that such that $r_1\boldsymbol{b}_1 + \ldots + r_4\boldsymbol{b}_4 + s\boldsymbol{e}_j = \boldsymbol{0}$. If $s = 0$, then the linear inependence of B is contradicted. Otherwise, we get

$$\boldsymbol{e}_j = (-r_1/s)\boldsymbol{b}_1 + \ldots + (-r_4/s)\boldsymbol{b}_4$$

which contradicts $\boldsymbol{e}_j \notin \text{sp}(B)$. The claim is now clear. ∎