MATH 232
First Midterm
October 6, 1999

ANSWER KEY

1.

Question:
Compute a vector $\boldsymbol{x} \in \mathbb{R}^3$ such that $3\boldsymbol{x} - [2, 1, -1] = [7, -2, 4]$.

Marking:
2 marks

Solution:

$$\begin{aligned} 3\boldsymbol{x} - [2, 1, -1] &= [7, -2, 4] \\ 3\boldsymbol{x} &= [9, -1, 3] \\ \boldsymbol{x} &= [3, -\frac{1}{3}, 1] \end{aligned}$$

2.

Question:
Write $\boldsymbol{c}$ as a linear combination of $\boldsymbol{r}$ and $\boldsymbol{b}$.
(See the text of the exam for the description of the vectors $\boldsymbol{c}$, $\boldsymbol{r}$ and $\boldsymbol{b}$.)

Marking:
3 marks for a correct solution
1 mark for correct notation (vector algebra)

4 marks total

Solution:
Let $\boldsymbol{a}$ denote the vector represented by the point A. Since R is the midpoint of OA, we have $\boldsymbol{a} = 2\boldsymbol{r}$. Since $OABC$ is a parallelogram, we have $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$. After substituting for $\boldsymbol{a}$ we obtain $\boldsymbol{b} = 2\boldsymbol{r} + \boldsymbol{c}$. Thus $\boldsymbol{c} = -2\boldsymbol{r} + \boldsymbol{b}$.

3.

Question:
(a) Find all values $c \in \mathbb{R}$ such that the vectors $[1, -4, 7]$ and $[-3, c, 5]$ are orthogonal.
(b) Compute the norm $\|[-3, 2, 4, 0, -2]\|$.
(c) Find a unit vector parallel to the vector $[-2, 1, 2, -4]$.

Marking:
(a), (b), (c) - 2 marks each

Solution:
(a) The vectors $[1, -4, 7]$ and $[-3, c, 5]$ are orthogonal iff $[1, -4, 7] \cdot [-3, c, 5] = 0$, that is, $-3 - 4c + 35 = 0$. The only value of c that satisfies this equation is $c = 8$.

(b) $\|[-3, 2, 4, 0, -2]\| = \sqrt{(-3)^2 + 2^2 + 4^2 + 0^2 + (-2)^2} = \sqrt{33}$.

(c) The norm of a unit vector is equal to 1. Two nonzero vectors are parallel iff one is a scalar multiple of the other one. For any scalar $r \in \mathbb{R}$ and any vector $\boldsymbol{v} \in \mathbb{R}^n$ we have $\|r\boldsymbol{v}\| = |r| \; \|\boldsymbol{v}\|$ (the homogeneity property of the norm). If $\|r\boldsymbol{v}\| = 1$, then $|r| = \frac{1}{\|\boldsymbol{v}\|}$, which implies $r = \frac{1}{\|\boldsymbol{v}\|}$ or $r = -\frac{1}{\|\boldsymbol{v}\|}$. Let $\boldsymbol{w} = [-2, 1, 2, -4]$ be the given vector. We have $\|\boldsymbol{w}\| = \sqrt{(-2)^2 + 1^2 + 2^2 + (-4)^2} = \sqrt{25} = 5$. Hence, the unit vectors parallel to $\boldsymbol{w}$ are $\frac{1}{5}\boldsymbol{w} = [-\frac{2}{5}, \frac{1}{5}, \frac{2}{5}, -\frac{4}{5}]$ and $-\frac{1}{5}\boldsymbol{w} = [\frac{2}{5}, -\frac{1}{5}, -\frac{2}{5}, \frac{4}{5}]$. Either of these two vectors is a sufficient answer to the question, since we were asked for "a vector" (not "all vectors").

4.

Question:
(a) Write down the augmented matrix of the system:

$$\begin{cases} 2x_1 - 2x_2 - \;\; 4x_3 = -6 \\ \;\; x_1 + 2x_2 + \;\; 4x_3 = \;\; 3 \\ \qquad\;\; 4x_2 + 11x_3 = \;\; 8 \end{cases}$$

(b) Convert the matrix from (a) to reduced row-echelon form by row operations.

Marking:
(a) - 2 marks
(b) - 3 marks

Solution:
(a)

$$\left[\begin{array}{rrr|r} 2 & -2 & -4 & -6 \\ 1 & 2 & 4 & 3 \\ 0 & 4 & 11 & 8 \end{array}\right]$$

(b) One possible sequence of elementary row operations is

$$\left[\begin{array}{ccc|c} 2 & -2 & -4 & -6 \\ 1 & 2 & 4 & 3 \\ 0 & 4 & 11 & 8 \end{array}\right] \underset{R_1 \to \frac{1}{2}R_1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 1 & 2 & 4 & 3 \\ 0 & 4 & 11 & 8 \end{array}\right] \underset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 3 & 6 & 6 \\ 0 & 4 & 11 & 8 \end{array}\right]$$
$$\underset{R_2 \to \frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & 11 & 8 \end{array}\right] \underset{R_3 \to R_3 - 4R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 0 \end{array}\right] \underset{R_3 \to \frac{1}{3}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$
$$\underset{R_2 \to R_2 - 2R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{R_1 \to R_1 + 2R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus the reduced row-echelon form of the matrix from part (a) is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

5.

Question:

Consider the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$, $\boldsymbol{b} \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2\times 4}$. It is given that the reduced row-echelon form of the augmented matrix $[A|\boldsymbol{b}]$ is the matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -3 & 9 \\ 0 & 0 & 1 & 7 & -4 \end{array}\right].$$

Find the general solution of the system $A\boldsymbol{x} = \boldsymbol{b}$ writing your final answer in vectorial form.

Marking:

3 marks

Solution:

From the first equation we find that $x_1 = 9 - 3x_2 + 3x_4$. From the second equation we find that $x_3 = -4 - 7x_4$. The variables x_2 and x_4 are free variables. The vectorial form of the general solution of the system $A\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \begin{bmatrix} 9 - 3x_2 + 3x_4 \\ x_2 \\ -4 - 7x_4 \\ x_4 \end{bmatrix}.$$

6.

Question:

Let A denote the matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

(a) Compute A^{-1}, the inverse matrix of A. (Give your answer in the form of a single 3×3 matrix.)
(b) Based on the elementary row operations that you used to answer the part (a), express the matrix A as a product of elementary matrices.

Marking:

(a) - 4 marks
(b) - 2 marks

Solution:

(a) We form the augmented matrix $[A|I]$ and transform the left half of the augmented matrix to the reduced row-echelon form by elementary row operations. One possible sequence of such elementary row operations is

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_1 \to R_1 - R_2}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R_1 \to \frac{1}{2}R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\underset{R_3 \to R_3 + 3R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{array}\right].$$

Therefore

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

(b) Let ρ_i ($i = 1, ..., k$) be the elementary row operations used in part (a). In our solution we have $k = 3$, but there may be more elementary row operations in other solutions. Let $E_i = \rho_i(I)$ be the elementary matrix corresponding to ρ_i ($i = 1, ..., k$). We have $I = E_k E_{k-1} \dots E_2 E_1 A$ and $A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$.
In our solution we have

$$\begin{aligned} \rho_1 &= R_1 \to R_1 - R_2 \\ \rho_2 &= R_1 \to \frac{1}{2} R_1 \\ \rho_3 &= R_3 \to R_3 + 3R_2. \end{aligned}$$

By applying the rules for computing inverses of elementary matrices we find

$$E_1^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Therefore one way of expressing A as a product of elementary matrices is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

7.

Question:

(a) Let $V = \{[3x - y, -x + y, 2x] \; : \; x, y \in \mathbb{R}\}$ be the set of all vectors $[3x - y, -x + y, 2x]$ where x, y are arbitrary real numbers. Decide whether V is a subspace of $\mathbb{R}^3$. Give a reason for your answer.

(b) Let $W = \{[-2x + y, y, 1, x - y] \; : \; x, y \in \mathbb{R}\}$ be the set of all vectors $[-2x + y, y, 1, x - y]$ where x, y are arbitrary real numbers. Decide whether W is a subspace of $\mathbb{R}^4$. Give a reason for your answer.

Marking:

(a) - 2 marks

(b) - 2 marks

Solution:

(a)

Answer: The set V is a subspace of $\mathbb{R}^3$.

Proof: Any of the following two arguments can be used:

1. $V = \{x[3, -1, 2] + y[-1, 1, 0] \; : \; x, y \in \mathbb{R}\} = \text{sp}([3, -1, 2], [-1, 1, 0])$. Therefore V is a subspace of $\mathbb{R}^3$ by the subspace property of a span (Theorem 1.14).

2. Verify that V is a non-empty subset of $\mathbb{R}^3$ closed under vector addition and also under scalar multiplication (Definition 1.16).

It is clear that V is a non-empty subset of $\mathbb{R}^3$.

Closure under vector addition: Let $\boldsymbol{u}, \boldsymbol{v} \in V$. (Remark: A common mistake was to take $\boldsymbol{u} = \boldsymbol{v}$. We must however prove the closure property for an arbitrary $\boldsymbol{u} \in V$ and an arbitrary $\boldsymbol{v} \in V$.) Denote $\boldsymbol{u} = [3x_u - y_u, -x_u + y_u, 2x_u]$, $\boldsymbol{v} = [3x_v - y_v, -x_v + y_v, 2x_v]$, where x_u, y_u, x_v, y_v are some real numbers. Then $\boldsymbol{u} + \boldsymbol{v} = [3(x_u + x_v) - (y_u + y_v), -(x_u + x_v) + (y_u + y_v), 2(x_u + x_v)]$ $= [3x_s - y_s, -x_s + y_s, 2x_s]$ where x_s and y_s are the real numbers defined by $x_s = x_u + x_v$ and $y_s = y_u + y_v$. Therefore $\boldsymbol{u} + \boldsymbol{v} \in V$.

Closure under scalar multiplication: Let $r \in \mathbb{R}$ and $\boldsymbol{v} \in V$. Denote $\boldsymbol{v} = [3x_v - y_v, -x_v + y_v, 2x_v]$, where x_v and y_v are some real numbers. Then $r\boldsymbol{v} = [3rx_v - ry_v, -rx_v + ry_v, 2rx_v] = [3x_t - y_t, -x_t + y_t, 2x_t]$ where x_t and y_t are the real numbers defined by $x_t = rx_v$ and $y_t = ry_v$. Therefore $r\boldsymbol{v} \in V$.

(b)
Answer: The set W is not a subspace of $\mathbb{R}^4$.
Proof: Any of the following three arguments can be used:
1. The set W does not contain the zero vector, but any subspace of $\mathbb{R}^n$ contains the zero vector $\mathbf{0} = [0, 0, \dots, 0]$.
2. The set W is not closed under vector addition: Let $\boldsymbol{u}, \boldsymbol{v} \in W$. The third coordinate of the vector sum $\boldsymbol{u} + \boldsymbol{v}$ is equal to $1 + 1 = 2$, and so $\boldsymbol{u} + \boldsymbol{v} \notin W$.
3. The set W is not closed under scalar multiplication: Let $r \in \mathbb{R}$ and $\boldsymbol{v} \in W$. The third coordinate of the scalar multiple $r\boldsymbol{v}$ is equal to r, and so $r\boldsymbol{v} \notin W$ whenever $r \neq 1$.

8.

Question:
Given is the following system of equations:

$$\left.\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 4x_3 & - & 3x_4 & = & 0 \\ 3x_1 & + & 7x_2 & - & 15x_3 & - & 8x_4 & = & 0 \end{array}\right\} (*)$$

Let W be the set of all solutions (regarded as column vectors) to this linear system. Find a basis for W.

Marking:
2 marks for finding the general solution of the system
2 marks for expressing the general solution as a span of two vectors
1 mark for verifying that the set of two vectors is a basis for its span

5 marks total

Solution:
First we solve the system by finding the reduced row-echelon form of its augmented matrix. Note that we are dealing with a homogeneous system and therefore we do not need to explicitly compute with the vector of the right-hand sides (which will remain equal to the zero vector throughout the Gauss reduction).

$$\begin{bmatrix} 1 & 2 & -4 & -3 \\ 3 & 7 & -15 & -8 \end{bmatrix} \underset{R_2 \to R_2 - 3R_1}{\sim} \begin{bmatrix} 1 & 2 & -4 & -3 \\ 0 & 1 & -3 & 1 \end{bmatrix} \underset{R_1 \to R_1 - 2R_2}{\sim} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & -3 & 1 \end{bmatrix}$$

Thus the general solution to the system (*) is

$$\boldsymbol{x} = \begin{bmatrix} -2x_3 + 5x_4 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

where x_3 and x_4 are free variables and can assume any real values. Therefore the set of solutions to the system (*) is equal to the span

$$\text{sp}\left(\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

Let $\boldsymbol{w}_1 = [-2, 3, 1, 0]^T$, $\boldsymbol{w}_2 = [5, -1, 0, 1]^T$. Let $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$ be a solution to (*), thus $\boldsymbol{x} \in \text{sp}(\boldsymbol{w}_1, \boldsymbol{w}_2)$. Suppose that $\boldsymbol{x} = c_1\boldsymbol{w}_1 + c_2\boldsymbol{w}_2$. By comparing the coordinates on both sides we conclude that $c_1 = x_3$, $c_2 = x_4$. In other words, any solution $\boldsymbol{x}$ is expressible *uniquely* as a linear combination of $\boldsymbol{w}_1, \boldsymbol{w}_2$. Therefore the set $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ is a basis for the set of all solutions to the system (*).

9.

Question:
Determine whether the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for the subspace of $\mathbb{R}^4$ spanned by this set, where

$$\boldsymbol{v}_1 = [2, 1, -3, 4], \ \boldsymbol{v}_2 = [1, -1, 0, 5], \ \boldsymbol{v}_3 = [1, 5, -6, -7].$$

Give a reason for your answer.

Marking:
3 marks for forming the 4×3 matrix and carrying out the Gauss reduction to the point (at least) when it is clear that there is a column without a pivot
2 marks for explaining why this calculation proves that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is not a basis for $\text{sp}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$.

5 marks total

Solution:
By Theorem 1.15, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\text{sp}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ iff, for any real numbers r_1, r_2, r_3,

$$r_1\boldsymbol{v}_1 + r_2\boldsymbol{v}_2 + r_3\boldsymbol{v}_3 = \boldsymbol{0} \qquad (*)$$

implies $r_1 = r_2 = r_3 = 0$. The equation (*) written in the matrix notation has the form

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 5 \\ -3 & 0 & -6 \\ 4 & 5 & -7 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (**)$$

We use the elementary row operations to reduce the coefficient matrix of (**) to the row-echelon form.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 5 \\ -3 & 0 & -6 \\ 4 & 5 & -7 \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & -1 & 5 \\ 2 & 1 & 1 \\ -3 & 0 & -6 \\ 4 & 5 & -7 \end{bmatrix}$$

$$\underset{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + 3R_1,\ R_4 \to R_4 - 4R_1}{\sim} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & -9 \\ 0 & -3 & 9 \\ 0 & 9 & -27 \end{bmatrix}$$

$$\underset{R_3 \to R_3 + R_2,\ R_4 \to R_4 - 3R_2}{\sim} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 3 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

At this point we see that there is no pivot in the third column, hence the variable r_3 is free and the linear system (**) has non-trivial solutions. Therefore the zero vector $\mathbf{0}$ can be expressed as non-trivial linear combinations (*) of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. Therefore the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is not a basis for sp$(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$.