Math 232, Elementary Linear Algebra

J. Hebron, Spring 2000

Mid-Term Examination #2

Wednesday, March 1st, 2000

Time: 50 minutes

Student ID Number

Name
(Please underline your family name)

Signature

Instructions:

- **Please fill-in the above information in ink.**
- **Do not open this exam until told to do so.**
- **No books, no notes, no calculators, no cell phones.**
- **Please sign the bottom of every page**
  (in case your exam becomes unstapled)

Question #:	1	2	3	4	5	6	7	8	9	10	Tot
Mark:											
Out of:	3	4	6	3	9	5	7	16	12	10	75

Part A (Problem 1): True or False?

Instructions: Indicate whether the following statements are true or false. No explanation required.

[mark]

1. (i) Let $\mathbf{S} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $\mathbf{S}$ is independent, then each vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of vectors in **S**. [1]

(a) true **(b)** false

(ii) If $\mathbf{H}$ is a row-echelon form of matrix $\mathbf{A}$, then the nonzero column vectors in $\mathbf{H}$ form a basis for the column space of $\mathbf{A}$. [1]

(a) true **(b)** false

(iii) The set $\mathbf{Q}$ of all rational numbers forms an infinite-dimensional vector space. [1]

(a) true **(b)** false

Part B (Problems 2 to 6): Short Answer

Instructions: Give a short answer/solution to each of the following questions/problems. No detailed explanation is required. It is assumed that any required work can be carried-out in your head, so it is not necessary to show your work. Questions having very short answers are provided with an answer box – please write the answer in this box if so provided.

[mark]

2. Suppose an m by n matrix $\mathbf{A}$ has a row space of dimension p.

(i) What is the dimension of **A**'s column space? [2]

answer

(ii) What is the dimension of **A**'s nullspace? [2]

answer

(Signature)

3. Let $\mathbf{A}$ be the following matrix:

$$\mathbf{A} = \begin{bmatrix} a_1 & r_1a_1 & r_2a_1 & \cdots & r_{n-1}a_1 & r_na_1 \\ a_2 & r_1a_2 & r_2a_2 & \cdots & r_{n-1}a_2 & r_na_2 \\ a_3 & r_1a_3 & r_2a_3 & \cdots & r_{n-1}a_3 & r_na_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m-1} & r_1a_{m-1} & r_2a_{m-1} & \cdots & r_{n-1}a_{m-1} & r_na_{m-1} \\ a_m & r_1a_m & r_2a_m & \cdots & r_{n-1}a_m & r_na_m \end{bmatrix}$$

where $a_1 \cdots a_m$ and $r_1 \cdots r_n$ are non-zero real numbers.

(i) What is the rank of $\mathbf{A}$? [2]

answer

(ii) Write down a basis for the row space of **A**. [2]

(iii) Write down a basis for the column space of **A**. [2]

4. If you consider differentiation to be a linear transformation $\mathbf{T}$, what is **T**'s nullspace? [3]

5. **(i)** Give an example of a vector space $\mathbf{V}$ which is not finitely generated. [3]
[*You should read parts (ii) and (iii) before answering this.*]

(ii) Let $\mathbf{W}$ be a subspace of $\mathbf{V}$, where $\mathbf{V}$ is given in **(i)**. Give an example of such a subspace which is itself not finitely generated. [3]

(iii) Let $\mathbf{W}'$ be a 3-dimensional subspace of $\mathbf{V}$, where $\mathbf{V}$ is given in **(i)**. Give an example of such a subspace. [3]

(Signature)

6. **(i)** What is the determinant of the matrix **A** given in problem #**3**? [2]

answer

(ii) What is the determinant of the following matrix? [3]

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & 0 & a_{3,3} & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & a_{4,4} & a_{4,5} \\ 0 & 0 & 0 & 0 & a_{5,5} \end{bmatrix}$$

answer

Part C (Problems 7 to 9): Show All Your Work

Instructions: Work out the following problems, showing all your work, and place the final answer in the answer box . Part marks will be awarded even if the final answer is wrong.

7. Evaluate the following determinant, where a to n and p are non-zero real numbers. [7]:

$$\begin{vmatrix} n & l & 0 & j & c \\ b & 0 & 0 & 0 & 0 \\ h & i & a & g & p \\ k & e & 0 & 0 & 0 \\ m & f & 0 & d & 0 \end{vmatrix}$$

Answer

(Signature)

8. Let $\mathbf{M}_2$ be the vector space of 2 by 2 matrices of real numbers. and let

$$\mathbf{B} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

form a basis for $\mathbf{M}_2$. What is $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in the basis **B**? [16]

answer

(Signature)

9. Let $\mathbf{F}$ be the vector space of functions $f:\Re \to \Re$, let $\mathbf{V}$ be the subspace of $\mathbf{F}$ spanned by the basis $\mathbf{B} = (\sin(x), \cos(x), 1)$, and let $\mathbf{V}'$ be the subspace of $\mathbf{F}$ spanned by the basis $\mathbf{B}' = (e^x, e^{-x})$. Define a linear transformation $\mathbf{T}: \mathbf{V} \to \mathbf{V}'$ as follows:

$$\mathbf{T}(\sin(x)) = \frac{e^x - e^{-x}}{2}$$

$$\mathbf{T}(\cos(x)) = \frac{e^x + e^{-x}}{2}$$

$$\mathbf{T}(1) = 0$$

(i) What is the matrix representation $\mathbf{A}$ of $\mathbf{T}$ relative to $\mathbf{B}, \mathbf{B}'$? [8]

A=

answer

(ii) What is the kernel of $\mathbf{T}$? [2]

(iii) Is the transformation invertible ? Explain. [2]

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Part D (Problem 10): Proof

10. Let **A** and **C** be matrices of real numbers, such that the matrix product **AC** is defined. Prove that the column space of **AC** is contained in the column space of **A** .

[10]

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