Math 232, Elementary Linear Algebra

J. Hebron, Spring 2000

# Mid-Term Examination #2

Wednesday, March 1st, 2000

Time: 50 minutes

| Student ID Number | Name (Please underline your family name) |
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| | SOLUTIONS |
| | Signature: J.H. |

**Instructions:**

- **Please fill-in the above information in ink.**
- **Do not open this exam until told to do so.**
- **No books, no notes, no calculators, no cell phones.**
- **Please sign the bottom of every page**
  (in case your exam becomes unstapled)

| Question #: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Tot |
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| Mark: | | | | | | | | | | | |
| Out of: | 3 | 4 | 6 | 3 | 9 | 5 | 7 | 16 | 12 | 10 | 75 |

**Part A (Problem 1): True or False?**

*Instructions: Indicate whether the following statements are true or false. No explanation required.*

**1. (i)** Let $\mathbf{S} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $\mathbf{S}$ is independent, then each vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of vectors in **S**. [ 1 ]

**(b) false: S** must also **span** $\mathbb{R}^n$ in order to be a basis for $\mathbb{R}^n$.

**(ii)** If **H** is a row-echelon form of matrix **A**, then the nonzero column vectors in **H** form a basis for the column space of **A**. [ 1 ]

**(b) false:** It is the column vectors of **A** corresponding to the nonzero column vectors in **H** which form a basis for the column space of **A**.

**(iii)** The set **Q** of all rational numbers forms an infinite-dimensional vector space. [ 1 ]

**(b) false: Q** is not closed under scalar multiplication, if you happen to multiply by an irrational number.

**Part B (Problems 2 to 6): Short Answer**

*Instructions: Give a short answer/solution to each of the following questions/problems. No detailed explanation is required. It is assumed that any required work can be carried-out in your head, so it is not necessary to show your work. Questions having very short answers are provided with an answer box – please write the answer in this box if so provided.*

**2.** Suppose an $m$ by $n$ matrix **A** has a row space of dimension $p$.

**(i)** What is the dimension of **A**'s column space? [ 2 ]

answer: p

**(ii)** What is the dimension of **A**'s nullspace? [ 2 ]

answer: n - p

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**3.** Let $\mathbf{A}$ be the following matrix:

$$\mathbf{A} = \begin{bmatrix} a_1 & r_1 a_1 & r_2 a_1 & \cdots & r_{n-1} a_1 & r_n a_1 \\ a_2 & r_1 a_2 & r_2 a_2 & \cdots & r_{n-1} a_2 & r_n a_2 \\ a_3 & r_1 a_3 & r_2 a_3 & \cdots & r_{n-1} a_3 & r_n a_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m-1} & r_1 a_{m-1} & r_2 a_{m-1} & \cdots & r_{n-1} a_{m-1} & r_n a_{m-1} \\ a_m & r_1 a_m & r_2 a_m & \cdots & r_{n-1} a_m & r_n a_m \end{bmatrix}$$

where $a_1 \cdots a_m$ and $r_1 \cdots r_n$ are non-zero real numbers.

**(i)** What is the rank of $\mathbf{A}$ ? [ 2 ]

| 1 |
|---|

answer

**(ii)** Write down a basis for the row space of **A**. [ 2 ]

$$\mathbf{B}_{row} = \left([1, r_1, r_2, \cdots, r_{n-1}, r_n]\right)$$

**(iii)** Write down a basis for the column space of **A**. [ 2 ]

$$\mathbf{B}_{col} = \left([a_1, a_2, a_3, \cdots, a_{m-1}, a_m]\right)$$

**4.** If you consider differentiation to be a linear transformation $\mathbf{T}$, what is **T**'s nullspace? [ 3 ]

The space of all constant functions.

**5.** **(i)** Give an example of a vector space $\mathbf{V}$ which is not finitely generated. [ 3 ]
[*You should read parts (ii) and (iii) before answering this.*]

eg. 1: The space $\mathbf{P}$ of all polynomials in $x$.
eg. 2: The space $\mathbf{F}$ of all functions $f: \Re \to \Re$

**(ii)** Let $\mathbf{W}$ be a subspace of $\mathbf{V}$, where $\mathbf{V}$ is given in **(i)**. Give an example of such a subspace which is itself not finitely generated. [ 3 ]

eg. 1: The space $\mathbf{P}_e$ of all polynomials composed of even powers of $x$.
eg. 2: The space $\mathbf{F}_e$ of all even functions $f: \Re \to \Re$ where $f(-x) = f(x)$

**(iii)** Let $\mathbf{W}'$ be a 3-dimensional subspace of $\mathbf{V}$, where $\mathbf{V}$ is given in **(i)**. Give an example of such a subspace. [ 3 ]

eg. 1: $\mathbf{W}' = sp(1, x, x^2)$
eg. 2: $\mathbf{W}' = sp(1, \cos(x), e^{x^2})$

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**6.** **(i)** What is the determinant of the matrix **A** given in problem #**3**? [ 2 ]

| 0 if n+1 = m otherwise undefined |
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answer

**(ii)** What is the determinant of the following matrix? [ 3 ]

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & 0 & a_{3,3} & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & a_{4,4} & a_{4,5} \\ 0 & 0 & 0 & 0 & a_{5,5} \end{bmatrix}$$

Answer:

$$a_{1,1}a_{2,2}a_{3,3}a_{4,4}a_{5,5}$$

## Part C (Problems 7 to 9): Show All Your Work

*Instructions: Work out the following problems, showing all your work, and place the final answer in the answer box . Part marks will be awarded even if the final answer is wrong.*

**7.** Evaluate the following determinant, where $a$ to $n$ and $p$ are non-zero real numbers. [ 7 ]:

$$\begin{vmatrix} n & l & 0 & j & c \\ b & 0 & 0 & 0 & 0 \\ h & i & a & g & p \\ k & e & 0 & 0 & 0 \\ m & f & 0 & d & 0 \end{vmatrix} = a \begin{vmatrix} n & l & j & c \\ b & 0 & 0 & 0 \\ k & e & 0 & 0 \\ m & f & d & 0 \end{vmatrix} = -ab \begin{vmatrix} l & j & c \\ e & 0 & 0 \\ f & d & 0 \end{vmatrix} = -abc \begin{vmatrix} e & 0 \\ f & d \end{vmatrix} = -abcde$$

Answer:

*-abcde*

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**8.** Let $\mathbf{M}_2$ be the vector space of 2 by 2 matrices of real numbers. and let

$$\mathbf{B}=\left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&-1\\0&0\end{bmatrix},\begin{bmatrix}1&-1\\0&3\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)$$

form a basis for $\mathbf{M}_2$. What is $\begin{bmatrix}1&-2\\3&4\end{bmatrix}$ in the basis **B**? [ 16 ]

Let $\mathbf{B}=\left(\vec{\mathbf{b}}_1,\vec{\mathbf{b}}_2,\vec{\mathbf{b}}_3,\vec{\mathbf{b}}_4\right)$, where the $\vec{\mathbf{b}}_j$ represent the above matrices.

Let $\vec{\mathbf{v}}=\begin{bmatrix}1&-2\\3&4\end{bmatrix}$.

Let $\mathbf{B}'=\left(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$, which is the usual basis for $\mathbf{M}_2$.

We can express the basis vectors $\vec{\mathbf{b}}_j$ in the $\mathbf{B}'$ basis as follows:

$\left(\vec{\mathbf{b}}_1\right)_{\mathbf{B}'}=[0,1,1,0]_{\mathbf{B}'}$, $\left(\vec{\mathbf{b}}_2\right)_{\mathbf{B}'}=[0,-1,0,0]_{\mathbf{B}'}$, $\left(\vec{\mathbf{b}}_3\right)_{\mathbf{B}'}=[1,-1,0,3]_{\mathbf{B}'}$, $\left(\vec{\mathbf{b}}_4\right)_{\mathbf{B}'}=[0,1,0,1]_{\mathbf{B}'}$

and the vector $\vec{\mathbf{v}}$ in the $\mathbf{B}'$ basis is: $(\vec{\mathbf{v}})_{\mathbf{B}'}=[1,-2,3,4]_{\mathbf{B}'}$.

We wish to solve $r_1\vec{\mathbf{b}}_1+r_2\vec{\mathbf{b}}_2+r_3\vec{\mathbf{b}}_3+r_4\vec{\mathbf{b}}_4=\vec{\mathbf{v}}$. This can be done in the $\mathbf{B}'$ basis by forming the augmented matrix and performing Gauss-Jordan reduction:

$$\left[\begin{array}{cccc|c}0&0&1&0&1\\1&-1&-1&1&-2\\1&0&0&0&3\\0&0&3&1&4\end{array}\right]\xrightarrow[\mathbf{R}_1\leftrightarrow\mathbf{R}_3]{}\left[\begin{array}{cccc|c}1&0&0&0&3\\1&-1&-1&1&-2\\0&0&1&0&1\\0&0&3&1&4\end{array}\right]\xrightarrow[\substack{\mathbf{R}_2\to\mathbf{R}_2-\mathbf{R}_1\\ \mathbf{R}_4\to\mathbf{R}_4-3\mathbf{R}_3}]{}\left[\begin{array}{cccc|c}1&0&0&0&3\\0&-1&-1&1&-5\\0&0&1&0&1\\0&0&0&1&1\end{array}\right]$$

$$\xrightarrow[\mathbf{R}_2\to\mathbf{R}_2+\mathbf{R}_3]{}\left[\begin{array}{cccc|c}1&0&0&0&3\\0&-1&0&1&-4\\0&0&1&0&1\\0&0&0&1&1\end{array}\right]\xrightarrow[\mathbf{R}_2\to\mathbf{R}_2-\mathbf{R}_4]{}\left[\begin{array}{cccc|c}1&0&0&0&3\\0&-1&0&0&-5\\0&0&1&0&1\\0&0&0&1&1\end{array}\right]$$

$$\xrightarrow[\mathbf{R}_2\to-\mathbf{R}_2]{}\left[\begin{array}{cccc|c}1&0&0&0&3\\0&1&0&0&5\\0&0&1&0&1\\0&0&0&1&1\end{array}\right]\qquad\therefore(\vec{\mathbf{v}})_{\mathbf{B}}=[3,5,1,1]_{\mathbf{B}}$$

| |
|---|
| $[\,3\,,5\,,1\,,1\,]_{\mathrm{B}}$ |

answer

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**9.** Let **F** be the vector space of functions $f:\Re \to \Re$, let **V** be the subspace of **F** spanned by the basis $\mathbf{B} = (\sin(x), \cos(x), 1)$, and let $\mathbf{V}'$ be the subspace of **F** spanned by the basis $\mathbf{B}' = \left(e^x, e^{-x}\right)$. Define a linear transformation $\mathbf{T}: \mathbf{V} \to \mathbf{V}'$ as follows:

$$\mathbf{T}(\sin(x)) = \frac{e^x - e^{-x}}{2}$$

$$\mathbf{T}(\cos(x)) = \frac{e^x + e^{-x}}{2}$$

$$\mathbf{T}(1) = 0$$

**(i)** What is the matrix representation **A** of **T** relative to $\mathbf{B}, \mathbf{B}'$ ? [ 8 ]

In the $\mathbf{B}'$ basis, we have:

$$T(\sin(x))_{\mathbf{B}'} = \left[\frac{1}{2}, -\frac{1}{2}\right]_{\mathbf{B}'}$$

$$T(\cos(x))_{\mathbf{B}'} = \left[\frac{1}{2}, \frac{1}{2}\right]_{\mathbf{B}'}$$

$$T(1)_{\mathbf{B}'} = [0, 0]_{\mathbf{B}'}$$

and therefore: $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

**(ii)** What is the kernel of **T** ? [ 2 ]

$\ker(\mathbf{T}) = sp(1)$

**(iii)** Is the transformation invertible ? Explain. [ 2 ]

No, for any of the following reasons:

- **A** is not a square matrix and therefore doesn't have an inverse.
- The kernel of **T** is not zero-dimensional.
- **V** and $\mathbf{V}'$ have different dimensions and therefore the transformation is not 1-to-1.

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**Part D (Problem 10): Proof**

**10.** Let **A** and **C** be matrices of real numbers, such that the matrix product **AC** is defined. Prove that the column space of **AC** is contained in the column space of **A** .

[ 10 ]

Let **C** be an *m* x *n* matrix. By definition, every vector $\vec{\mathbf{v}}$ in the column space of **AC** is a linear combination of columns of **AC** and therefore it is of the form $\vec{\mathbf{v}} = (\mathbf{AC})\vec{\mathbf{x}}$ for some $\vec{\mathbf{x}} \in \Re^n$. However, $(\mathbf{AC})\vec{\mathbf{x}} = \mathbf{A}(\mathbf{C}\vec{\mathbf{x}})$ by the associative law of matrix multiplication, so $\vec{\mathbf{v}} = \mathbf{A}\vec{\mathbf{w}}$ where $\vec{\mathbf{w}} = \mathbf{C}\vec{\mathbf{x}}$ . Therefore, $\vec{\mathbf{v}}$ can be expressed as a linear combination of the column vectors of **A** . In other words, $\vec{\mathbf{v}}$ is in the column space of **A** . Since $\vec{\mathbf{v}}$ can be any vector in the column space of **AC** , this shows that the column space of **AC** is contained in the column space of **A** .

**QED**

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