DEPARTMENT OF MATHEMATICS AND STATISTICS

Second Midterm Key

MATH 232 March 3, 1999

11:30 am – 12:20 pm

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[3] **1.** Let $F : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear transformation which satisfies

$$F([-1,3]) = [-4,6,2,0], \qquad F([2,-1]) = [3,-2,1,5]$$

Compute the standard matrix representation of F.

ANSWER

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

EXPLANATION Note that

$$[1,0] = \frac{1}{5}[-1,3] + \frac{3}{5}[2,-1] \qquad [0,1] = \frac{2}{5}[-1,3] + \frac{1}{5}[2,-1]$$

Therefore, since F is linear, we have

$$F([1,0]) = \frac{1}{5}F([-1,3]) + \frac{3}{5}F([2,-1]) = [1,0,1,3]$$
$$F([0,1]) = \frac{2}{5}F([-1,3]) + \frac{1}{5}F([2,-1]) = [-1,2,1,1]\,.$$

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[6] **2.** On the separate sheet circulated with the exam you will find the definition of a *vector space over* $\mathbb{R}$.

Let V be a vector space over $\mathbb{R}$.

From the axioms listed prove that, for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in V,

$$\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{a}+\boldsymbol{c} \ \textbf{ implies } \ \boldsymbol{b} = \boldsymbol{c}\,.$$

ANSWER Suppose that $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{a}+\boldsymbol{c}$. Then

$$\begin{aligned}
\boldsymbol{b} &= \boldsymbol{0}+\boldsymbol{b} && \text{by A3} \\
&= (\boldsymbol{a}+(-\boldsymbol{a}))+\boldsymbol{b} && \text{by A4} \\
&= (\boldsymbol{b}+\boldsymbol{a})+(-\boldsymbol{a}) && \text{by A1, A2} \\
&= (\boldsymbol{c}+\boldsymbol{a})+(-\boldsymbol{a}) && \text{hypothesis} \\
&= (\boldsymbol{a}+(-\boldsymbol{a}))+\boldsymbol{c} && \text{by A1, A2} \\
&= \boldsymbol{0}+\boldsymbol{c} && \text{by A4} \\
&= \boldsymbol{c} && \text{A3.}
\end{aligned}$$

■

[3] **3.** (a) Let V be a vector space over $\mathbb{R}$. Let W be a subset of V.

State necessary and sufficient conditions for W to be a subspace of V.

ANSWER W is a subspace of V if and only if

1. $W \neq \emptyset$.
2. W is closed under vector addition, i.e., $\boldsymbol{a} + \boldsymbol{b} \in W$ for all $\boldsymbol{a}$ and $\boldsymbol{b}$ in W.
3. W is closed under scalar multiplication, i.e., $r\boldsymbol{a} \in W$ for all r in $\mathbb{R}$ and $\boldsymbol{a}$ in W.

[4] (b) Let V be the vector space ${}^{\mathbb{R}}\mathbb{R}$ of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let W denote the set

$$\left\{f \in {}^{\mathbb{R}}\mathbb{R} : (\forall x,\, y \in \mathbb{R})\,[xy > 0 \text{ implies } f(x) = f(y)]\right\}$$

Decide whether W is a subspace of V and justify your answer.

ANSWER Yes

We check the three conditions mentioned in (a).

1. $W \neq \emptyset$ because every constant function is in W.
2. Consider $f,\, g \in W$ and $x,\, y \in \mathbb{R}$ such that $xy > 0$. From the definition of W,
$$f(x) = f(y) \text{ and } g(x) = g(y)\,.$$
Therefore
$$(f+g)(x) = f(x) + g(x) = f(y) + g(y) = (f+g)(y)$$
and so $f + g \in W$. Thus W is closed under vector addition.
3. Consider $r \in \mathbb{R}$, $f \in W$ and $x,\, y \in \mathbb{R}$ such that $xy > 0$. From the definition of W, $f(x) = f(y)$. Therefore
$$(rf)(x) = r \cdot f(x) = r \cdot f(y) = (rf)(y)\,.$$
and so $rf \in W$. Thus W is closed under scalar multiplication.

[2] **4.** (a) Let V be a vector space over $\mathbb{R}$, and $\boldsymbol{v}$ be a vector in V.
Let $\mathcal{B} = \langle \boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \rangle$ be an ordered basis of V.

Define the coordinate vector $\boldsymbol{v}_{\mathcal{B}}$ of $\boldsymbol{v}$ with respect to $\mathcal{B}$.

> ANSWER
>
> $\boldsymbol{v}_{\mathcal{B}}$ is defined to be $[r_1, \ldots, r_n]$, where $r_1, \ldots, r_n$ are the unique real numbers satisfying
> $$\boldsymbol{v} = r_1\boldsymbol{b}_1 + \ldots + r_n\boldsymbol{b}_n\,.$$

[3] (b) Let $\mathcal{B}$ denote the ordered basis $\langle [-1, 1, 2], [1, -1, 0], [0, 1, 0] \rangle$ for $\mathbb{R}^3$.

Compute the coordinate vector of $[1, 0, 0]$ with respect to $\mathcal{B}$.

> ANSWER
> $$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

EXPLANATION

The required coordinate vector $\boldsymbol{x}$ satisfies:

$$x_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This system can be solved in one's head because, looking at the third component we see that $x_1 = 0$. Then, looking at the first component, we have $x_2 = 1$. Finally, $x_3 = 1$ by inspection of the second component. ■

5. Let V be the subspace of $\mathbb{R}[x]$ consisting of all polynomials of degree at most 1, and W be the subspace of $\mathbb{R}[x]$ consisting of all polynomials of degree at most 2.

Let $\mathcal{B}$, $\mathcal{C}$ denote the ordered bases

$$\langle x,\, 1\rangle, \quad \langle x^2+x+1,\, x+1, 1\rangle$$

for V, W respectively.

Let $T : V \to W$ denote the linear transformation defined by $T(p) = (x+1)p$.

[5] (a) **Compute the matrix which represents T with respect to $\mathcal{B}$, $\mathcal{C}$.**

ANSWER

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

[2] (b) **Is there a linear transformation $T' : W \to V$ such that $T' \circ T$ is the identity on V? Justify your answer.**

ANSWER

Yes. By a theorem there is a unique linear transformation $T' : W \to V$ which satisfies $T'(x^2+x) = x$, $T'(x+1) = 1$, $T'(1) = 0$. It is clear that $T' \circ T(x) = x$ and $T' \circ T(1) = 1$. Since $T' \circ T$ agrees with id_V on a basis of V, $T' \circ T$ is the identity on V. ■

EXPLANATION

(a) $T(x) = (x+1)x = x^2 + x = 1(x^2+x+1) + 0(x+1) + (-1)1$

$T(1) = (x+1)1 = x+1 = 0(x^2+x+1) + 1(x+1) + 0(1)$

[2] **6.** (a) Let $\boldsymbol{b}$, $\boldsymbol{c}$ be vectors in $\mathbb{R}^3$.

Explain the relationship of the vector $b \times c$ to the vectors b, c in geometrical terms.

ANSWER

The norm $\|\boldsymbol{b} \times \boldsymbol{c}\|$, i.e., the length of $\boldsymbol{b} \times \boldsymbol{c}$ is the area of the rectangle determined by $\boldsymbol{b}$ and $\boldsymbol{c}$.

$\boldsymbol{b} \times \boldsymbol{c}$ is orthogonal to the plane determined by $\boldsymbol{b}$ and $\boldsymbol{c}$.

$\langle \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{b} \times \boldsymbol{c} \rangle$ is a right-handed triple of vectors.

[2] (b) **Find the area of the triangle whose vertices are the points $(1,1,3)$, $(0,1,0)$, $(1,1,0)$ in $\mathbb{R}^3$.**

ANSWER

$3/2$

EXPLANATION

(b) Subtracting $(0,1,0)$ from all three points translates the given triangle to the triangle $(1,0,3)$, $(0,0,0)$, $(1,0,0)$, i.e., to the triangle determined by $\mathbf{0}$, $(1,0,3)$, and $(1,0,0)$.

The area of the triangle $\mathbf{0}\,(1,0,3)\,(1,0,0)$ is half of the area of the parallelogram determined by $(1,0,3)$, $(1,0,0)$, which is

$$\|[1,0,3] \times [1,0,0]\| = \|[0,3,0]\| = 3\,. \qquad \blacksquare$$

[2] **7.** (a) **Evaluate the determinant**

$$\begin{vmatrix} a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \\ a+4 & a+6 & a+9 \end{vmatrix}$$

ANSWER

-3

EXPLANATION Applying the row operations $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\begin{vmatrix} a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \\ a+4 & a+6 & a+9 \end{vmatrix} = \begin{vmatrix} a+1 & a+4 & a+7 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+1 & a+4 & 3 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$$

where the second equality comes from the column operation $C_3 \to C_3 - C_2$.

Expanding down column 3, gives the result. ■

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[2] (b) Let A be an $n \times n$ matrix.

Define the matrix adj(A) called the adjoint of A.

State the most important property of the adjoint of A.

ANSWER

Definition

adj(A) is the $n \times n$ matrix whose (i,j)-th entry is $(-1)^{i+j}|A_{ji}|$, where A_{ji} is the (j,i)-th minor of A.

Most important property

$$A\,\text{adj}(A) = \text{adj}(A)\,A = |A|I\,.$$

[2] **8.** (a) **Express the following complex number in the form** $a + ib$**, where** $a,\, b \in \mathbb{R}$ **and** $i^2 = -1$

$$\frac{(2-i)(1+i)}{1+2i}\,.$$

ANSWER

$1 - i$

[2] (b) **Compute** $r,\, \theta \in \mathbb{R}$**, such that** $r > 0$**,** $0 \le \theta < 2\pi$**, and**

$$2 - i = r\, e^{i\theta}\,.$$

ANSWER

$r = \sqrt{5}$

$\theta = 2\pi - \arctan(1/2)$

ROUGH WORK