MATH 232
Second Midterm
November 17, 1999

ANSWER KEY

1.

Question:
Given are four vectors $\boldsymbol{x} = [0, 1, 1, 0]$, $\boldsymbol{y} = [1, -2, 5, -1]$, $\boldsymbol{z} = [2, 1, 2, 3]$ and $\boldsymbol{w} = \boldsymbol{x} + \boldsymbol{z} = [2, 2, 3, 3]$. It is given that the set $B = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ is a basis for the subspace V of $\mathbb{R}^4$.
(a) Let $A \in \mathbb{R}^{4\times 4}$ and let the row vectors of A be $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}$. Find the rank of A. Give a reason for your answer.
(b) Find a basis B' for V such that $\boldsymbol{w} \in B'$. Justify your answer.

Marking:
(a) - 2 marks
(b) - 1 mark for a correct answer
2 marks for justifying the answer

3 marks total for part (b)

5 marks total for Question 1—parts (a) and (b)

Solution:
(a) Since $B = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ is a basis for V and $\boldsymbol{w} \in V$, we see that the row space of A is precisely V. By Theorem 2.4, rank(A) = dim(rowspace(A)) = dim$(V) = 3$.

A more labourious solution to part (a) consists in reducing A to a row-echelon form H and counting the number of non-zero rows in H.

(b) There are infinitely many correct answers. While looking for B' we can take advantage of Theorem 2.3 (part 3), which implies that any set of 3 vectors that spans V (or any set of 3 independent vectors in V) is a basis for V.
For example, from $\boldsymbol{w} = \boldsymbol{x} + \boldsymbol{z}$ it follows that, for any scalars $r_1, r_2, r_3 \in \mathbb{R}$, we have

$$\begin{aligned} r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{w} &= (r_1 + r_3)\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z} \\ r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z} &= (r_1 - r_3)\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{w} \end{aligned}$$

and so sp$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})$ = sp$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = V$. Therefore $B' = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}\}$ is a basis for V.

A more labourious solution to part (b) consists in setting up the matrix C that has $\boldsymbol{w}$ as the first column and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ (in some order) as the remaining three columns, and reducing C to a row-echelon form H. The columns of C corresponding to the columns of H containing pivots then form a basis for V. Since H will have a pivot in the first column, this will yield a basis for V that contains $\boldsymbol{w}$.

2.

Question:
Let $\boldsymbol{t} = [-5, 2]$ and $\boldsymbol{u} = [3, -1]$. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(\boldsymbol{t}) = [1, -3, 2]$ and $T(\boldsymbol{u}) = [-1, 2, 0]$.
(a) Find the standard matrix representation of T.
(b) Use the answer to part (a) to compute $T([2, 1])$.

Marking:
(a) - 3 marks
(b) - 1 mark

Solution:
(a) Using the Corollary on p. 146, the standard matrix representation of T is the matrix $[\,T(\boldsymbol{e}_1)\ T(\boldsymbol{e}_2)\,]$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the standard basis for $\mathbb{R}^2$. Therefore we need to express $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as linear combinations of $\boldsymbol{t}$ and $\boldsymbol{u}$. Using Gauss reduction we find

$$\left[\begin{array}{rr|r} -5 & 3 & 1 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{rr|r} -5 & 3 & 0 \\ 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right].$$

(It is wiser to perform both reductions in one process.) Therefore $\boldsymbol{e}_1 = \boldsymbol{t} + 2\boldsymbol{u}$, $\boldsymbol{e}_2 = 3\boldsymbol{t} + 5\boldsymbol{u}$ and

$$\begin{aligned} T(\boldsymbol{e}_1) &= T(\boldsymbol{t}) + 2T(\boldsymbol{u}) = [1, -3, 2] + 2[-1, 2, 0] = [-1, 1, 2] \\ T(\boldsymbol{e}_2) &= 3T(\boldsymbol{t}) + 5T(\boldsymbol{u}) = 3[1, -3, 2] + 5[-1, 2, 0] = [-2, 1, 6]\,. \end{aligned}$$

The standard matrix representation of T is

$$[\,T(\boldsymbol{e}_1)\ T(\boldsymbol{e}_2)\,] = \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 2 & 6 \end{bmatrix}.$$

(b) We compute

$$\begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 10 \end{bmatrix}.$$

Thus $T([2, 1]) = [-4, 3, 10]$.

3.

Question:
Let F be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. We say that $f \in F$ is an *odd function* if $f(-x) = -f(x)$ for every $x \in \mathbb{R}$. Let S denote the set of all odd functions. Decide whether S is a subspace of F. Justify your answer.

Marking:
5 marks

Solution:
The set S is a subspace of F. We will prove this using Theorem 3.2.
1. The set S is nonempty. For example, the constant function $f(x) = 0$ is in the set S.
2. The set S is closed under vector addition: Suppose $f_1, f_2 \in S$. Then

$$(f_1 + f_2)(-x) = f_1(-x) + f_2(-x) = -f_1(x) + (-f_2(x)) = -(f_1(x) + f_2(x)) = -(f_1 + f_2)(x)$$

and so $f_1 + f_2 \in S$.
3. The set S is closed under scalar multiplication: Let $c \in \mathbb{R}$, $f \in S$. Then

$$(cf)(-x) = c(f(-x)) = c(-f(x)) = -cf(x) = -(cf)(x)$$

and so $cf \in S$.

4.

Question:
Let P_n be the vector space of all polynomials in x, with real coefficients and of degree less than or equal to n, together with the zero polynomial. Let

$$\mathcal{B} = (x^3 + x,\ x^2 - x,\ x - 1,\ 1) \quad \text{and} \quad \mathcal{B}' = (x^2,\ x,\ 1)$$

be ordered bases for P_3 and P_2, respectively. Let the linear transformation $T : P_3 \to P_2$ be defined by $T(p) = p'$, the derivative of p with respect to x.
(a) Find the matrix representation of T relative to $\mathcal{B}$, $\mathcal{B}'$.
(b) Find the coordinate vector $(x^3 - x)_{\mathcal{B}}$ of $x^3 - x$ relative to $\mathcal{B}$.

Marking:
(a) - 4 marks
(b) - 2 marks

Solution:
(a) We have to find $T(\boldsymbol{b})_{\mathcal{B}'}$ for all $\boldsymbol{b} \in \mathcal{B}$. We compute

$$\begin{aligned}
T(x^3 + x)_{\mathcal{B}'} &= (3x^2 + 1)_{\mathcal{B}'} = [3, 0, 1] \\
T(x^2 - x)_{\mathcal{B}'} &= (2x - 1)_{\mathcal{B}'} = [0, 2, -1] \\
T(x - 1)_{\mathcal{B}'} &= 1_{\mathcal{B}'} = [0, 0, 1] \\
T(1)_{\mathcal{B}'} &= 0_{\mathcal{B}'} = [0, 0, 0]\,.
\end{aligned}$$

By Definition 3.11 the matrix representation of T relative to $\mathcal{B}$, $\mathcal{B}'$ is

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}.$$

(b) By Definition 3.8 the coordinate vector $(x^3 - x)_{\mathcal{B}}$ is the vector $[r_1, r_2, r_3, r_4] \in \mathbb{R}^4$ such that

$$x^3 - x = r_1 \cdot (x^3 + x) + r_2 \cdot (x^2 - x) + r_3 \cdot (x - 1) + r_4 \cdot 1 .$$

Because of the particular form of $\mathcal{B}$ (namely, $\mathcal{B}$ contains exactly one polynomial of degree i for each $i = 3, 2, 1, 0$), it is easy to find the r_i's. We compute

$$\begin{aligned} x^3 - x &= 1 \cdot (x^3 + x) - 2x \\ &= 1 \cdot (x^3 + x) + 0 \cdot (x^2 - x) + (-2) \cdot (x - 1) - 2 \\ &= 1 \cdot (x^3 + x) + 0 \cdot (x^2 - x) + (-2) \cdot (x - 1) + (-2) \cdot 1 . \end{aligned}$$

Thus

$$(x^3 - x)_{\mathcal{B}} = [1, 0, -2, -2] .$$

5.

Question:
Given are four points $P = (1, 1, 3)$, $Q = (2, 0, 5)$, $R = (1, 4, 1)$ and $S = (3, 2, 5)$. Decide whether P, Q, R and S are coplanar (i.e. whether they all lie in one plane in $\mathbb{R}^3$). Give a reason for your answer.

Marking:
1 mark for computing the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$
1 mark for noting that the points are coplanar iff the volume of the corresponding box is 0
1 mark for using the correct formula for the volume of the box
1 mark for correctly computing the determinant

4 marks total

Solution:
Let us consider the vectors

$$\begin{aligned} \boldsymbol{a} &= [2, 0, 5] - [1, 1, 3] = [1, -1, 2] \\ \boldsymbol{b} &= [1, 4, 1] - [1, 1, 3] = [0, 3, -2] \\ \boldsymbol{c} &= [3, 2, 5] - [1, 1, 3] = [2, 1, 2] . \end{aligned}$$

The points P, Q, R and S are coplanar if and only if the volume of the box (parallelepiped) determined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is zero. This volume is the absolute value of the determinant $\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{vmatrix}$. We compute

$$\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} = 0 .$$

Thus the points P, Q, R and S are coplanar.

6.

<u>Question:</u>
Evaluate the determinant

$$\begin{vmatrix} 2 & 4 & 7 & -1 & 4 \\ 0 & -3 & 2 & -1 & 4 \\ 1 & -13 & 3 & -2 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 0 & 0 \end{vmatrix}.$$

<u>Marking:</u>
4 marks

<u>Solution:</u>

$$\begin{vmatrix} 2 & 4 & 7 & -1 & 4 \\ 0 & -3 & 2 & -1 & 4 \\ 1 & -13 & 3 & -2 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 0 & 0 \end{vmatrix} = 3 \cdot (-1)^{5+2} \cdot \begin{vmatrix} 2 & 7 & -1 & 4 \\ 0 & 2 & -1 & 4 \\ 1 & 3 & -2 & 3 \\ 0 & 1 & 2 & 1 \end{vmatrix} = -3 \cdot \begin{vmatrix} 0 & 1 & 3 & -2 \\ 0 & 2 & -1 & 4 \\ 1 & 3 & -2 & 3 \\ 0 & 1 & 2 & 1 \end{vmatrix}$$

$$= -3 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -1 & 3 \end{vmatrix} = -3 \cdot \begin{vmatrix} -7 & 8 \\ -1 & 3 \end{vmatrix}$$

$$= -3 \cdot ((-7) \cdot 3 - 8 \cdot (-1)) = -3 \cdot (-21 + 8) = 39$$

7.

<u>Question:</u>
Let $A \in \mathbb{R}^{n \times n}$. Use the known facts about determinants to prove that A is invertible if and only if 0 is not an eigenvalue of A.

<u>Marking:</u>
4 marks

<u>Solution:</u>
By Theorem 4.3, A is invertible if and only if $\det(A) \neq 0$, that is, if and only if $\det(A - 0I) \neq 0$, which is the case if and only if 0 is not an eigenvalue of A.

8.

Question:
(a) Let $A \in \mathbb{R}^{n \times n}$ and assume that A is diagonalizable. Explain briefly how you can compute, for any positive integer k, the power A^k using the diagonalization of A.
(b) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Use diagonalization to find the formula for A^k, $k \geq 1$.

Marking:
(a) - 2 marks
(b) - 6 marks

8 marks total

Solution:
(a) Let $A \in \mathbb{R}^{n \times n}$. By Definition 5.3, A is diagonalizable if there exists an invertible matrix C such that $C^{-1}AC = D$, a diagonal matrix. If this is the case, then $A = CDC^{-1}$ and

$$A^k = CD^kC^{-1}.$$

(b) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We have

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 1 \cdot 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$$

and so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Since the eigenvalues are distinct, A is diagonalizable by Theorem 5.3. (Remark: The fact that A is diagonalizable follows immediately from Theorem 5.5, but to actually find the diagonalization we of course need to compute the eigenvalues and eigenvectors.) By Gauss reduction we find

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus the eigenvectors corresponding to λ_1 are the nonzero vectors in sp$([-1,1])$ and the eigenvectors corresponding to λ_2 are the nonzero vectors in sp$([1,1])$. Let us select the eigenvectors $\boldsymbol{v}_1 = [-1,1]$ and $\boldsymbol{v}_2 = [1,1]$. If we let

$$C = [\boldsymbol{v}_1 \ \boldsymbol{v}_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

then the diagonalization of A is $C^{-1}AC = D$ and by part (a)—see also Corollary 2 on p. 307—we have $A^k = CD^kC^{-1}$. To compute the inverse of C we use the adjoint matrix formula or Gauss reduction. We find

$$C^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

and so

$$\begin{aligned} A^k &= CD^kC^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2^k \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix}. \end{aligned}$$