NEWTON POLYGONS, SUCCESSIVE MINIMA, AND DIFFERENT BOUNDS FOR DRINFELD MODULES OF RANK 2

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Abstract. Let $K = \mathbb{F}_q(T)$. For a Drinfeld $A$-module $\phi$ of rank 2 defined over $C_\infty$, there is an associated exponential function $e_\phi$ and lattice $\Lambda_\phi$ in $C_\infty$ given by uniformization over $C_\infty$. We explicitly determine the Newton polygons of $e_\phi$ and the successive minima of $\Lambda_\phi$. When $\phi$ is defined over $K_\infty$, we give a refinement of Gardeyn’s bounds for the action of wild inertia at $\infty$ on the torsion points of $\phi$, and a criterion for the lattice field to be unramified over $K_\infty$. If $\phi$ is in addition defined over $K$, we make explicit Gardeyn’s bounds for the action of wild inertia at finite primes on the torsion points of $\phi$, using results of Rosen, and this gives an explicit bound on the degree of the different divisor of division fields of $\phi$ over $K$.

1. Introduction

Let $K = \mathbb{F}_q(T)$, $A = \mathbb{F}_q[T]$, where $q$ is a power of a prime $p$, and suppose that $\infty = (\frac{1}{T})$ is the place at infinity of $K$ with associated normalized valuation function $v = v_\infty : K \to \mathbb{Z} \cup \{+\infty\}$. Let $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the completion of $K$ at $\infty$, and $C_\infty$ be the completion of an algebraic closure of $K_\infty$. Denote also by $v$ the extension of $v$ from $K$ to $C_\infty$. Let the absolute value associated to $v$ be given by $|x| = q^{-v(x)}$.

For a specified homomorphism $\iota : A \to F$, where $F$ is a field, a Drinfeld $A$-module $\phi$ over $F$ is a homomorphism $\phi : A \to F(\tau)$ such that for all $a \in A$, $\phi_a := \phi(a)$ has constant term $\iota(a)$, where $\tau : z \mapsto z^q$ is the $q$-th power Frobenius endomorphism and $F(\tau)$ denotes the ring of twisted polynomials over $F$ satisfying $\tau \alpha = \alpha^q \tau$ for all $\alpha \in F$. We require that the image of $\phi$ not be contained in $F$. It can be shown there

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is an integer \( r \geq 1 \) called the rank of \( \phi \) such that
\[
\phi_a = \sum_{i=0}^{\deg a} a_i(\phi, a) \tau^i
\]
for all \( a \in A \). Note that a Drinfeld \( A \)-module of rank \( r \) is completely determined by its value \( \phi_T = T + \sum_{i=1}^{r} a_i(\phi) \tau^i \).

For a Drinfeld \( A \)-module \( \phi \) of rank 2 over \( K \), one knows by uniformization (cf. [10]) that there is an \( A \)-lattice \( \Lambda_\phi = \Lambda_{\phi, \infty} \subseteq C_\infty \) of rank 2 and a surjective analytic function \( e_\phi = e_{\phi, \infty} : C_\infty \to C_\infty \) with zero set equal to \( \Lambda_\phi \) and such that
\[
e_\phi(a z) = \phi_a \circ e_\phi(z)
\]
for all \( a \in A \) and normalized so the derivative \( de_\phi(z)/dz \) of \( e_\phi(z) \) is equal to 1. The function \( e_\phi(z) \) is called the exponential function associated to \( \phi \). It is uniquely determined by the above properties and can be written in the form \( e_\phi(z) = \sum_{i=0}^{\infty} c_i \tau^i(z) \) where \( \tau(z) = z^q, c_i \in C_\infty \), and \( c_0 = 1 \).

Let \( \phi \) be a Drinfeld \( A \)-module over a field \( F \) with respect to a specified homomorphism \( \iota : A \to F \). For any \( a \in A, a \neq 0 \), we define the \( A \)-module of \( a \)-torsion points of \( \phi \) as
\[
\phi[a] = \{ \lambda \in \Phi \mid \phi_a(\lambda) = 0 \},
\]
and let \( F(\phi[a]) \) be the field obtained by adjoining the \( a \)-torsion points of \( \phi \) to \( F \) (here \( \Phi \) denotes a fixed algebraic closure of \( F \)).

There has been some interest in studying the field \( K(\phi[a]) \) generated over \( K \) by the \( a \)-torsion points of \( \phi \) [2, 3, 4, 5, 11, 12, 13, 14, 17]. A natural object which arises in bounding the ramification over \( \infty \) is the field \( K_\infty(\Lambda_{\phi, \infty}) \) which contains the field generated by the \( a \)-torsion points of \( \phi \) over \( K_\infty \). Since \( \Lambda_{\phi, \infty} \) is the zero set of the analytic function \( e_{\phi, \infty}(z) \), the differnt of \( K_\infty(\Lambda_{\phi, \infty})/K_\infty \) can be bounded using information from the Newton polygon of \( e_{\phi, \infty}(z) \) [4].

In this paper, we explicitly determine the Newton polygon and slopes of \( e_{\phi, \infty}(z) \) for a general Drinfeld \( A \)-module \( \phi \) of rank 2 defined over \( K \) (in fact, over \( C_\infty \)) determined by \( \phi_T = T + a_1(\phi) \tau + a_2(\phi) \tau^2 \). The different cases of Newton polygons which arise depend on \( v(j(\phi)) \), where \( j(\phi) \) is the \( j \)-invariant of \( \phi \), defined by \( j(\phi) = a_1(\phi)^{q+1}/a_2(\phi) \). Some applications of this determination of the Newton polygons are given.

We point out that the essential nature of the Newton polygon of \( e_\phi \) depends only on the \( C_\infty \)-isomorphism class of \( \phi \). Suppose we have an isomorphism \( f \) from \( \phi \) to \( \phi' \), where \( \phi_T = T + a_1 \tau + a_2 \tau^2 \), and \( \phi'_T = T + a'_1 \tau + a'_2 \tau^2 \). It follows that \( f \circ \phi_a = \phi'_a \circ f \) for all \( a \in A \), \( f(z) = cz \) for some \( c \in C_\infty^* \), and \( a'_i = a_i/c^{q-1} \). Using Equation (1), we see by induction that \( e_{\phi'}(z) = \sum_{i=0}^{\infty} c_i \tau^i(z) = \sum_{i=0}^{\infty} a_i/c^{q-1} \tau^i(z) \). It follows that \( f \circ e_\phi = e_{\phi'} \circ f \) and hence \( \Lambda_{\phi'} = c \Lambda_\phi \). Thus, the slopes of the Newton polygon of \( e_{\phi'} \) are simply the

\[2\]
slopes of the Newton polygon of $e_\phi$ translated by $-v(c)$, with projected sides having the same lengths and coordinates.

We give explicit bounds on the ramification of $K_\infty(\Lambda_{\phi, \infty})/K_\infty$ at $\infty$ based on a method which slightly refines the one in [4]. The ingredient which is needed to make the bound explicit is based on work of Gekeler, which relates $v(j(z))$ and $v(z)$ for $z \in F$, where $F = \{z \in C_\infty : |z| = |z|_i = \inf_{x \in K_\infty} |z - x| \geq 1\}$. In the case of $v(j(\phi)) \geq -q$, the bound does not depend on $\phi$.

Finally, using work of Rosen [15], we make explicit Gardeyn’s bounds for the action of wild inertia at finite primes on the torsion points of $\phi$. As a result, we obtain an explicit bound on the degree of the different divisor of division fields of $\phi$ over $K$.

2. Newton polygon of the exponential function associated to a twist of $\phi$

We say $\Lambda \subseteq C_\infty$ is an $A$-lattice of rank $r$ if $\Lambda = A\lambda_1 + \ldots + A\lambda_r$ with $\lambda_i \in C_\infty$ being $K_\infty$-linearly independent, and we refer to $\{\lambda_1, \ldots, \lambda_r\}$ as an $A$-basis for $\Lambda$.

Let $B_\kappa = \{\lambda \in \Lambda : |\lambda| \leq \kappa\}$ for $\kappa \in \mathbb{R}$. We define the $i$th successive minimum $\nu_i$ to be the minimum of the set of $\kappa$ such that $B_\kappa$ contains $i$ number of $K$-linearly independent elements for $i = 1, 2, \ldots, r$, and $(\nu_1, \ldots, \nu_r)$ is called the successive minima of $\Lambda$. An ordered $A$-basis $(\lambda_1, \ldots, \lambda_r)$ for $\Lambda$ is called a minimal $A$-basis for $\Lambda$ if $|\lambda_i| = \nu_i$ for $i = 1, 2, \ldots, r$ (note: a minimal ordered $A$-basis for $\Lambda$ always exists because of the discreteness of the valuations of elements in $\Lambda$).

**Lemma 2.1.** If $\{\omega_1, \ldots, \omega_n\}$ is an $A$-basis for an $A$-lattice $\Lambda$ such that $|\omega_1| \leq |\omega_2| \leq \cdots \leq |\omega_n|$, then the following are equivalent.

- $(\omega_1, \ldots, \omega_n)$ is a minimal $A$-basis for $\Lambda$
- $|\sum_{i=1}^n a_i \omega_i| = \max \{|a_i \omega_i| : 1 \leq i \leq n\}$ for all $a_i \in A$.

**Proof.** [18, Lemma 4.2] □

By uniformization, we may regard the coefficients $c_i = c_i(z)$ of $e_\phi$ as functions on the upper-half plane $\Omega$, where $\Omega := C_\infty - K_\infty$, and $\phi$ is the Drinfeld $A$-module associated to the $A$-lattice $\Lambda = A + Az$. These functions are dubbed the *para-Eisenstein series* by Gekeler [7], and are studied in [6, 8].

Let

$$\mathcal{F} = \{z \in C_\infty : |z| = |z|_i \geq 1\},$$

$$\mathcal{F}_k = \{z \in C_\infty : |z| = |z|_i = q^k\},$$
where \( k \geq 0 \) and \( |z|_i = \inf_{x \in K_{\infty}} |z - x| \).

The subset \( \mathcal{F} \) of \( \Omega \) is a kind of fundamental domain for \( \Omega \) under the action of \( \Gamma = \text{GL}_2(A) \) in the following sense.

**Proposition 2.2.** Each element \( z \in \Omega \) is \( \Gamma \)-equivalent to an element of \( \mathcal{F} \).

**Proof.** [6, Corollary 6.7]. \( \square \)

Further properties of \( \mathcal{F} \) can be found in loc. cit.

**Theorem 2.1.** For \( z \in \mathcal{F} \) we have the following

- \( \log_q |j(z)| \leq q \iff z \in \mathcal{F}_0 \)
- Suppose \( k \geq 1 \). Then \( \log_q |j(z)| = q^{k+1} \iff z \in \mathcal{F}_k \).
- Suppose \( m \geq 1 \). Then \( m - 1 < -v(z) < m \) if and only if \( q^m < -v(j(z)) < q^{m+1} \). Furthermore, we have the following linear interpolation property,

\[
-v(z) = \frac{-v(j(z)) - q^m}{q^m(q-1)} + m - 1.
\]

**Proof.** [9, Corollary 3.11, Remark 2.4]. \( \square \)

**Lemma 2.3.** If \( z \in \mathcal{F} \), then \( (1, z) \) is a minimal \( A \)-basis for the \( A \)-lattice \( \Lambda = A + Az \).

**Proof.** Let \( z \in \mathcal{F} \). Assume that \( (1, z) \) is not a \( A \)-minimal basis for \( A + Az \). Since \( |z| \geq 1 \), by Lemma 2.1, there exist \( a, b \in A \) such that

\[
|az + b| < \max\{|az|, |b|\} \text{ and } |az| = |b|.
\]

It is clear that a must be nonzero. Thus \( |z + b/a| < |z| \) and \( |z| = |b/a| \). Since \( z \) is in \( \mathcal{F} \), we have \( |z| = |z|_i = \inf_{x \in K_{\infty}} |z - x| \). Therefore, \( |z| \leq |z + b/a| \). This contradicts the inequality above. \( \square \)

Let \( \phi \) be a Drinfeld \( A \)-module of rank 2 over \( C_{\infty} \). Let \( \Lambda = \Lambda_\phi \) be the \( A \)-lattice associated to \( \phi \) by uniformization, which is the zero set of the exponential function \( e_\phi \). We know by the fundamental domain property of \( \mathcal{F} \) (cf. Proposition 2.2) that \( \phi \) is isomorphic over \( C_{\infty} \) to another Drinfeld \( A \)-module \( \phi' \) such that its exponential function \( e_{\phi'} \) has associated \( A \)-lattice \( \Lambda' = A + Az \) where \( z \in \mathcal{F} \).

It suffices to determine the Newton polygon of \( e_{\phi'} \) as the Newton polygon of \( e_\phi \) can be deduced from that of \( e_{\phi'} \) (see Section 3 for further details).

By Lemma 2.3, \( (1, z) \) is a minimal \( A \)-basis for \( \Lambda' = A + Az \).
2.1. Case 1: \(-v(j(\phi)) \leq q\). By Theorem 2.1, this corresponds to the situation \(z \in \mathcal{F}_0\). The following argument can be found in [6, p. 513]. Since \(|z| = 1\), we have that \(|az + b| = \max \{|a|, |b|\}\) holds for any \(a, b\) in \(A\) by Lemma 2.1. This implies \(\Lambda'\) has precisely \(q^{2(i+1)}\) elements of valuation \(\geq -i\), and for each \(i \geq 0\), there are \(q^{2(i+1)} - q^{2i}\) elements of valuation \(-i\).

Hence, the Newton polygon of \(e_{\phi'}\) has one segment of length \(q^{2i}(q^2 - 1)\) of slope \(i\) for each \(i \geq 0\).

2.2. Case 2 (i): \(q < -v(j(\phi)) \neq q^{m+1}\) for any \(m \geq 1\). By Theorem 2.1, this corresponds to the situation \(q^{m-1} < |z| < q^m\) for some \(m \geq 1\). Let \(|z| = q^\kappa\) where \(m - 1 < \kappa < m\). Furthermore, we know that \(0 < \kappa = -v(z) = \frac{-v(j(\phi)) - q^m}{q^m(q-1)} + m - 1\).

Now, by Lemma 2.1

\[|az + b| = \max \{q^\kappa |a|, |b|\}\]

holds for any \(a, b\) in \(A\). In this situation, we see that there are two types of non-zero elements of \(\Lambda'\), those with valuation in \(-\mathbb{Z}_{\geq 0}\) and those with valuation in \(-(\kappa + \mathbb{Z}_{\geq 0})\).

Thus, the possible slopes of segments of the Newton polygon of \(e_{\phi'}\), in order, are

\[0, 1, \ldots, m - 1, \kappa, m, \kappa + 1, m + 1, \kappa + 2, m + 2, \ldots\]

We now have to count the number of elements of \(\Lambda'\) of each possible valuation using (2).

For \(0 \leq i \leq m - 1\), there are \(q^{i+1} - q^i\) elements of valuation \(-i\). For \(j \geq 0\), there are \(q^{m+2j+1} - q^{m+2j}\) elements of valuation \(-(\kappa+j)\). For \(j \geq 0\), there are \(q^{m+2j+2} - q^{m+2j+1}\) elements of valuation \(-(m+j)\).

Hence, the Newton polygon of \(e_{\phi'}\) has one segment of length \(q^{i+1} - q^i\) of slope \(i\) for each \(0 \leq i \leq m - 1\); one segment of length \(q^{m+2j+1} - q^{m+2j}\) of slope \(\kappa+j\) followed by one segment of length \(q^{m+2j+2} - q^{m+2j+1}\) of slope \(m+j\), for each \(j \geq 0\).

2.3. Case 2 (ii): \(-v(j(\phi)) = q^{m+1}\) for some \(m \geq 1\). By Theorem 2.1, this corresponds to the situation \(|z| = q^m\). Thus, as \(|az + b| = \max \{q^m |a|, |b|\}\) holds for any \(a, b\) in \(A\), there are \(q^{i+1}\) elements of \(\Lambda'\) of valuation \(\geq -i\) if \(0 \leq i \leq m - 1\) and \(q^{2i-m+2}\) elements of \(\Lambda'\) of valuation \(\geq -i\) if \(i \geq m\). In particular, for \(0 \leq i \leq m - 1\), there are \(q^{i+1} - q^i\) elements of valuation \(-i\), and for \(i \geq m\), there are \(q^{2i-m+2} - q^{2i-m}\) elements of valuation \(-i\).

Hence, the Newton polygon of \(e_{\phi'}\) has one segment of length \(q^i(q-1)\) of slope \(i\) for each \(0 \leq i < m\) and one segment of length \(q^{2i-m}(q^2 - 1)\) of slope \(i\) for each \(i \geq m\).
3. Newton polygon of $e_\phi$ and valuations of successive minima

Recall $\phi$ is a given Drinfeld $A$-module of rank 2 over $C_\infty$ with associated exponential function $e_\phi$ and $A$-lattice $\Lambda = \Lambda_\phi$.

The calculation in Section 2 determined the Newton polygon of $e_{\phi'}$ where $\phi'$ was a Drinfeld $A$-module isomorphic to $\phi$ over $C_\infty$ such that its associated $A$-lattice is of the form $\Lambda' = A + Az$ with $z \in \mathcal{F}$.

Since $\Lambda' = c\Lambda$ for some $c \in C_\infty^*$, we know the slopes of the Newton polygon of $e_\phi$ are obtained by translating the slopes of the Newton polygon of $e_{\phi'}$ by $v(c)$.

Let $\phi_T = T + a_1\tau + a_2\tau^2$, $\phi'_T = T + a'_1 + a'_2\tau^2$, $e_\phi = \sum_i c_i\tau^i$, and $e_{\phi'} = \sum_i c'_i\tau^i$. We have that $a'_1 = a_1/c^{q-1}$, $c'_i = c_i/c^{q-1}$, and set $c_0 = c'_0 = 1$ as a normalization.

We know that the first two vertices of the Newton polygon of $e_\phi$ are either $(1, 0), (q, v(c_1))$ in Case 2 or $(1, 0), (q^2, v(c_2))$ in Case 1. In all cases, the slope of the first segment of the Newton polygon of $e_{\phi'}$ was 0.

In Case 2, we have that $v(c_1) = v(a_1) + q$ since $c_1 = a_1/Tq - T$, so the slope of the first segment of the Newton polygon of $e_\phi$ is $s_1 = v(a_1 + q)/q - 1$. The slopes of the remaining segments are those of $e_{\phi'}$ translated by $s_1$.

On the other hand, in Case 1, we have that $v(c_2) = v(a_2) + q^2$; this is because $c_2 = a_1c_1^2 + a_2$ and $v(a_2) < v(a_1c_1^2)$ in this case. Hence, the slope of the first segment of the Newton polygon of $e_\phi$ is $s_1 = v(a_2 + q^2)/q^2 - 1$. The slopes of the remaining segments are those of $e_{\phi'}$ translated by $s_1$.

Putting everything together, we obtained the following theorem.

**Theorem 3.1.** Let $\phi$ be a Drinfeld $A$-module of rank 2 over $C_\infty$ given by $\phi_T = T + a_1\tau + a_2\tau^2$. Let $e_\phi$ be its associated exponential function. Let $m$ be the least positive integer such that $-v(j(\phi)) \leq q^{m+1}$. Let $s_1 = v(a_2 + q^2)/q^2 - 1$ in Case 1 and $s_1 = v(a_1 + q)/q - 1$ in Case 2. Then the Newton polygon of $e_\phi$ is determined as follows.

**Case 1:** $-v(j(\phi)) \leq q$

The Newton polygon of $e_\phi$ has one segment of length $q^2(q^2 - 1)$ of slope $i + s_1$ for each $i \geq 0$.

**Case 2 (i):** $q < -v(j(\phi)) \neq q^{m+1}$

Let $\kappa = -v(j(\phi))/q^{m(q-1)} + m - 1$. The Newton polygon of $e_\phi$ has one segment of length $q^{i+1} - q^i$ of slope $i + s_1$ for each $0 \leq i \leq m - 1$; one segment of length $q^{m+2j+1} - q^{m+2j}$ of...
slope $\kappa + j + s_1$ followed by one segment of length $q^{m+2j+2} - q^{m+2j+1}$ of slope $m+j+s_1$, for each $j \geq 0$.

Case 2 (ii): $q < -v(j(\phi)) = q^{m+1}$

The Newton polygon of $e_\phi$ has one segment of length $q^i(q-1)$ of slope $i+s_1$ for each $0 \leq i < m$ and one segment of length $q^{2n-m}(q^2-1)$ of slope $i+s_1$ for each $i \geq m$.

Corollary 3.1. Assume the hypotheses and notation of Theorem 3.1. Let $\Lambda$ be the $A$-lattice associated to $\phi$ by uniformization, and suppose $(\lambda_1, \lambda_2)$ is a minimal $A$-basis for $\Lambda$. Then $v(\lambda_1) = -s_1 = v(\lambda_2)$ if $-v(j(\phi)) \leq q$, and $v(\lambda_1) = -s_1, v(\lambda_2) = -s_1 - \kappa$ if $q < -v(j(\phi)) \leq q^{m+1}$.

Proof. Recall from the proof of Theorem 3.1, that $\Lambda' = A + Az = c\Lambda$ for some $z \in \mathcal{F}$, where $c \in C^*_\infty$, and $(1, z)$ is a minimal $A$-basis for $\Lambda'$. Furthermore, it was shown that $v(c) = s_1$.

Now, $(\lambda'_1, \lambda'_2) = (\frac{1}{c}, \frac{1}{z})$ is a choice of minimal $A$-basis for $\Lambda$ since $(1, z)$ is a minimal $A$-basis for $\Lambda' = c\Lambda$. Therefore, $v(\lambda'_1) = -v(c) = -s_1$, and $v(\lambda'_2) = -v(c) + v(z) = -s_1$ in Case 1, and $-s_1 - \kappa$ in Case 2. □

Corollary 3.2. Assume the hypotheses and notation of Theorem 3.1 and further that $\phi$ is defined over $K_\infty$. Let $\Lambda$ be the $A$-lattice associated to $\phi$ by uniformization.

If $-v(j(\phi)) \leq q$, then $K_\infty(\Lambda)/K_\infty$ is unramified if and only if $v(a_2) \equiv -1 \pmod{q^2-1}$.

If $q^m < -v(j(\phi)) \leq q^{m+1}$ for $m \geq 1$, then $K_\infty(\Lambda)/K_\infty$ is unramified if and only if $v(a_1) \equiv -1 \pmod{q-1}$ and $v(j(\phi)) \equiv -q^m \pmod{q^m(q-1)}$.

Remark 3.3. Let $z \in \mathcal{F}_k$, where $k \geq 0$, and let $\phi$ be the Drinfeld $A$-module associated to the $A$-lattice $A + Az$, where $\phi_T = T + a_1(z)\tau + a_2(z)\tau^2$. The proof of Theorem 3.1 can also be used to give a different proof of the special case $k \leq 0$ of [8, Theorem 2.13], which determines the valuations $v(a_i(z))$ in terms of $k$ (and $v(j(z))$ if $k = 0$). The idea is to take $\phi = \phi'$ in the proof of Theorem 3.1 so $c = 1$, but we also know that $v(c) = s_1$, which shows that $v(a_2(z)) = -q^2$ in Case 1 and $v(a_1(z)) = -q$ in Case 2. Note however, both proofs require Gekeler’s theory as a fundamental ingredient.

The case $k = 0$ of [8, Theorem 2.13], together with results in [1], can be used to determine the valuations $v(c_i(z))$ in some new cases not covered in [6].

4. Gardeyn’s bounds for wild ramification at $\infty$

Let $\phi$ be a Drinfeld $A$-module of rank 2 defined over $K_\infty$, $e_\phi$ its associated exponential function, and $\Lambda_\phi = \Lambda_{\phi,\infty}$ its associated $A$-lattice in $C_\infty$. 

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In the following theorem, we give explicit bounds on the ramification of \( K_\infty(\Lambda_\phi)/K_\infty \) based on a slight refinement of the method in [4], which we present in the specific case of rank 2. We point out that the upper bound of the following theorem is not optimal as Corollary 3.2 shows.

**Theorem 4.1.** Let \( \phi \) be a Drinfeld \( A \)-module of rank 2 over \( K_\infty \) and let \( \mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty) \) be the different of \( K_\infty(\Lambda_\phi)/K_\infty \). Let \( m \) be the least positive integer such that \( -v(j(\phi)) \leq q^m \). Then

\[
v(\mathcal{D}(K_\infty(\Lambda_\phi)/K_\infty)) \leq \begin{cases} 
1 & \text{if } -v(j(\phi)) \leq q \\
1 + \kappa (q^{m+1} - 1) & \text{if } q < -v(j(\phi)) \leq q^{m+1}
\end{cases}
\]

where \( \kappa = -\frac{v(j(\phi)) - q^m}{q^m(q-1)} + m - 1 \).

*Proof.* Put \( \Lambda = \Lambda_\phi \), and let \( (\lambda_1, \lambda_2) \) be a minimal \( A \)-basis for \( \Lambda \) such that \( z = \lambda_2/\lambda_1 \in \mathcal{F} \).

For any \( s > 0 \), there exist \( d \in C_s^2 \) and \( \delta \) such that \( v(d) = -v(\lambda_2) + \delta \), where \( 0 \leq \delta < \frac{1}{q-1} \), and the ramification index of \( K_\infty'(d) \) divides \( q^s - 1 \).

Let \( \Lambda^0 = A\lambda_1^0 + A\lambda_2^0 \), where \( \lambda_i^0 = d\lambda_i \). Then \( (\lambda_1^0, \lambda_2^0) \) is a minimal \( A \)-basis for \( \Lambda^0 \) since \( (\lambda_1, \lambda_2) \) is a minimal \( A \)-basis for \( \Lambda \), and \( K_\infty'(\Lambda^0) = K_\infty'(\Lambda) \). Let \( G_{\Lambda^0} \) be the Galois group of \( K_\infty'(\Lambda^0)/K_\infty' \), and \( K_\infty^0 \) be the maximal tamely ramified subextension of \( K_\infty'(\Lambda^0)/K_\infty' \), corresponding to the Sylow \( p \)-subgroup \( P_{\Lambda^0} \) of \( G_{\Lambda^0} \) (recall that \( p = \text{char}(K) \)). For \( \sigma \in G_{\Lambda^0} \), \( \sigma\lambda_i^0 = \alpha\lambda_i^0 \), where \( \alpha \in \mathbb{F}_q^* \), as \( |\sigma\lambda_i^0| = |\lambda_i^0| \). It follows that \( \lambda_i^0 \in K_\infty^0 \) and \( K_\infty'(\Lambda^0) = K_\infty(\lambda_i^0) \).

Now, \( v(\lambda_2^0) = \delta \geq 0 \). Using [16, III Cor. 2, p. 66], we have that

\[
\mathcal{D}(K_\infty(\lambda_2^0)/K_\infty^0) = \prod_{\sigma \in P_{\Lambda^0}, \sigma \neq 1} (\sigma\lambda_2^0 - \lambda_2^0).
\]

For \( \sigma \in P_{\Lambda^0} \) with \( \sigma \neq 1 \), we have that \( \sigma\lambda_2^0 = \beta\lambda_2^0 + \lambda_2^0 \), where \( \beta \) is nonzero in \( A \) satisfies \( |\beta| \leq |\lambda_2/\lambda_1| \) by Lemma 2.1. It follows that \( \#P_{\Lambda^0} \leq q |\lambda_2/\lambda_1| \). Finally, \( |\sigma\lambda_2^0 - \lambda_2^0| = |\beta\lambda_2^0| \geq |\lambda_2^0| \) so \( v(\sigma\lambda_2^0 - \lambda_2^0) \leq v(\lambda_2^0) \), and hence

\[
v(\mathcal{D}(K_\infty(\lambda_2^0)/K_\infty^0)) \leq (q |\lambda_2/\lambda_1| - 1) v(\lambda_2^0) = (q |\lambda_2/\lambda_1| - 1) (v(\lambda_1/\lambda_2) + \delta).
\]

The extension \( K_\infty^0/K_\infty \) is tamely ramified, so we obtain that

\[
v(\mathcal{D}(K_\infty(\Lambda)/K_\infty)) \leq 1 + v(\mathcal{D}(K_\infty(\lambda_2^0)/K_\infty^0)).
\]
From Corollary 3.1, if \(-v(j(\phi)) \leq q\), then \(|z| = 1\) and \(\delta = 0\), and if \(q < -v(j(\phi)) \leq \beta q^{m+1}\), then \(|z| = q^\kappa\). Thus, we have

\[
v(D(K_\infty(\Lambda)/K_\infty)) = \begin{cases} 
1 & \text{if } -v(j(\phi)) \leq q \\
1 + (\kappa + \delta)(q^{\kappa+1} - 1) & \text{if } q < -v(j(\phi)) \leq \beta q^{m+1},
\end{cases}
\]

where \(0 \leq \delta < \frac{1}{q^m-1}\). Taking \(s \to \infty\) gives the desired bound. \(\square\)

**Remark 4.1.** For Theorem 4.1, using [4] directly would instead yield a bound of 1 if \(-v(j(\phi)) \leq q\), and \(1 + 2\kappa q^{\kappa+1}\) if \(q < -v(j(\phi)) \leq \beta q^{m+1}\).

We notice that in the range of \(v(j(\phi)) \geq -q\), the bound on the different of \(K_\infty(\Lambda_\phi)/K_\infty\) does not depend on \(\phi\).

5. **Gardeyn’s bounds for wild ramification at finite primes \(p\)**

Let \(p\) be a finite prime of \(K\), \(K_p\) be the completion at \(p\), and denote by \(v_p\) its associated valuation. It is well-known that \(\phi\) has potentially Tate (resp. potentially good) reduction over \(K_p\) if \(v_p(j(\phi)) < 0\) (resp. \(v_p(j(\phi)) \geq 0\)), and that the stable reduction occurs over a finite tamely ramified extension of \(K_p\) [15, Lemma 5.2].

By [4], we have that

\[
v_p(D(K_p(\phi[a])/K_p)) \leq \begin{cases} 
2v_p(a) & \text{if } \phi \text{ has good reduction over } K_p, \\
2v_p(a) + 1 & \text{if } \phi \text{ has good reduction over } K_p', \\
2v_p(a) + 1 - 2v_p(\lambda_1) & \text{if } \phi \text{ has Tate reduction over } K_p',
\end{cases}
\]

where \(K_p'\) is a finite tamely ramified extension of \(K_p\), and \(\lambda_1\) is defined as follows:

In the case that \(\phi\) has Tate reduction over \(K_p'\), we obtain from uniformization that there is a Drinfeld \(A\)-module \(\psi\) of rank 1 and a surjective exponential function \(e_{\phi,p} : C_p \to C_p\) such that \(e_{\phi,p} \circ \psi_a = \phi_a \circ e_{\phi,p}\) for all \(a \in A\), where \(C_p\) is the completion of an algebraic closure of \(K_p\).

The zeroes of \(e_{\phi,p}\) form a \(A\)-lattice \(\Lambda_p = \Lambda_{\phi,p}\) of rank 1 in \(C_p\), so suppose \(\Lambda_p = A\lambda_1\). Note it is necessarily the case that \(v(\lambda_1) < 0\) and \((\lambda_1)\) is a minimal \(A\)-basis for \(\Lambda_p\).

From [15, Lemma 5.3], we have that \(v_p(\lambda_1) = \frac{1}{q-1}v_p(j(\phi))\).

Combining the above estimations yields the following explicit upper bound for the different \(D(K_p(\phi[a])/K_p)\).
Theorem 5.1. Let $\phi$ be a Drinfeld $A$-module of rank 2 over $K_p$ and let $D(K_p(\phi[a])/K_p)$ be the different of $K_p(\phi[a])/K_p$. Then
\[
v_p(D(K_p(\phi[a])/K_p)) \leq \begin{cases} 
2v_p(a) & \text{if } \phi \text{ has good reduction over } K_p, \\
2v_p(a) + 1 & \text{if } v_p(j(\phi)) \geq 0 \text{ and } \phi \text{ has bad reduction over } K_p, \\
2v_p(a) + 1 - \frac{2}{q-1}v_p(j(\phi)) & \text{if } v_p(j(\phi)) < 0.
\end{cases}
\]

For a finite extension $L/K$, let $D(L/K)$ be the different divisor of $L/K$ and define the degree with respect to $K$ of $D(L/K)$ as
\[
\deg_K D(L/K) = \sum_v \max \{ v(D(L_w/K_v)) : w \mid v \} \deg_K v,
\]
where $v$ ranges through all normalized places of $K$, $w$ through all places of $L$ lying over each $v$, and $D(L_w/K_v)$ is the different of $L_w/K_v$. It can be shown that
\[
\deg_L D(L/K) \leq n' \deg_K D(L/K),
\]
where $n'$ is the geometric extension degree of $L/K$. Since $K_\infty(\phi[a]) \subseteq K_\infty(\Lambda_{\phi,\infty})$, we obtain:

Theorem 5.2. Let $\phi$ be a Drinfeld $A$-module of rank 2 over $K$, and let $D(K(\phi[a])/K)$ be the different divisor of $K(\phi[a])/K$. Then
\[
\deg_K D(K(\phi[a])/K) \leq 2\deg_K a + \deg_K \eta + \frac{2}{q-1}\deg_K \delta
\]
\[+ v_\infty(D(K_\infty(\Lambda_{\phi,\infty})/K_\infty)) \]
where $\delta$ is the (monic) denominator of $j(\phi)$ as represented by a fraction in reduced form, and $\eta$ is the product of finite primes $p$ such that $\phi$ has bad reduction over $K_p$.

This combined with Theorem 4.1 gives an explicit bound on $\deg_K D(K(\phi[a])/K)$ in terms of $j(\phi)$, the primes of bad reduction of $\phi$, and $a$.

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References


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