Abstract

The \( n \)-card problem is to determine the minimal intervals \([u,v]\) such that for every \( n \times n \) stochastic matrix \( A \) there is an \( n \times n \) permutation matrix \( P \) (depending on \( A \)) such that \( \text{tr}(PA) \in [u,v] \). This problem is closely related to classical mathematical problems from industry and management, including the linear assignment problem and the travelling salesman problem. The minimal intervals for the \( n \)-card problem are known only for \( n \leq 4 \).

We introduce a new method of analysis for the \( n \)-card problem that makes repeated use of the Extreme Principle. We use this method to answer a question posed by Sands, by showing that \([1,2]\) is a solution to the \( n \)-card problem for all \( n \geq 2 \). We also show that each closed interval of length \( \frac{n}{n-1} \) contained in \([0,2)\) is a solution to the \( n \)-card problem for all \( n \geq 2 \).

1 Introduction

Let \( n \geq 2 \) be an integer. An \( n \times n \) stochastic matrix is an \( n \times n \) matrix \((a_{ij})\) of non-negative real numbers, each of whose row sums is 1. A transversal sum of \((a_{ij})\) is a sum of the form \( \sum_{i=1}^{n} a_{\sigma(i),i} \), for some permutation \( \sigma \) of \( \{1,2,\ldots,n\} \). A solution to the \( n \)-card problem is an interval \([u,v]\) such that every \( n \times n \) stochastic matrix contains at least one transversal sum in \([u,v]\). Equivalently, a solution to the \( n \)-card problem is an interval \([u,v]\) such that for every \( n \times n \) stochastic matrix \( A \) there is an \( n \times n \) permutation matrix \( P \) such that \( \text{tr}(PA) \in [u,v] \). We wish to determine the minimal solutions to the \( n \)-card problem for each \( n \), namely those solutions \([u,v]\) for which no proper subinterval of \([u,v]\) is a solution.

The \( n \)-card problem is closely related to well-known mathematical problems with industrial and management applications, involving the possible values of \( \text{tr}(PA) \) for a permutation matrix \( P \) and a fixed square matrix \( A \). In particular, the linear assignment problem, “one of the most famous problems in linear programming and in combinatorial optimization [BDM09],” is to minimize \( \text{tr}(PA) \) over all permutation matrices \( P \) (see [BDM09, Chapter 4] for a detailed historical account of the development of algorithms for its solution, and an equivalent formulation in terms of weighted bipartite matchings); the travelling salesman problem is the special case in which the permutation corresponding to \( P \) is cyclic [Flo56].

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The terminology of the \( n \)-card problem reflects its original formulation \cite{San01, LS05} involving a set of \( n \) cards, each containing \( n \) non-negative real numbers written in a row and summing to 1, with the transversal sum representing the diagonal sum formed when the cards are placed one below the other according to some permutation. In 2001, Sands \cite{San01} asked for a proof that \( \left[ \frac{1}{2}, \frac{3}{2} \right] \) is a solution to the 3-card problem. Lenza and Sands \cite{LS05} introduced the generalization to the \( n \)-card problem in 2005.

The interval \( [0,1] \) is a minimal solution to the \( n \)-card problem for all \( n \geq 2 \) \cite[Lemma 3]{San11}, but it is not the only minimal solution. Indeed, it is easily checked by hand that the minimal solutions to the 2-card problem are
\[
[0,1], \quad [1,2].
\] (1.1)

The minimal solutions to the 3-card problem are \cite{LS05, San11}
\[
[0,1], \quad \left[ \frac{1}{2}, \frac{3}{2} \right], \quad [1,2],
\] (1.2)

and the minimal solutions to the 4-card problem are \cite{LS05, San11}
\[
[0,1], \quad \left[ \frac{1}{3}, \frac{4}{3} \right], \quad \left[ \frac{2}{3}, \frac{5}{3} \right], \quad [1,2].
\] (1.3)

The method of \cite{LS05} and \cite{San11} is to establish a particular interval \([u,v]\) as a solution to the \( n \)-card problem (for \( n = 3 \) or \( n = 4 \)), by using intersecting permutations to show that the number of transversal sums greater than \( v \), plus the number of transversal sums less than \( u \), is always less than \( n! \). The method has two drawbacks: it relies on a laborious case analysis for \( n = 4 \), and does not extend to \( n \geq 5 \) \cite[p.6]{LS05}.

In this paper we introduce a new method for analysing the \( n \)-card problem that makes repeated use of the Extreme Principle \cite{Zei07}. We believe that this method could shed light on other problems involving \( \text{tr}(PA) \), where \( P \) is a permutation matrix and \( A \) is a fixed square matrix. The Extreme Principle directs attention to the “largest” and “smallest” elements of a problem. In the present context, we assume for a contradiction that no transversal sum of an \( n \times n \) stochastic matrix lies in some interval \([u,v]\), and then consider the smallest transversal sum \( d \) greater than \( v \). Then, if a transversal sum is less than \( d \), it must be less than \( u \). We seek such transversal sums, involving exactly \( n-2 \) of the original summands of \( d \), in order to reach a contradiction. We thereby obtain strong new restrictions for all \( n \geq 5 \). In particular, we solve Problem 5 of \cite{San11} as follows.

**Theorem 1.1.** For all \( n \geq 2 \), the interval \([1,2]\) is a solution to the \( n \)-card problem.

We also prove the following result.

**Theorem 1.2.** For all \( n \geq 2 \), each closed interval of length \( \frac{n}{n-1} \) contained in \([0,2]\) is a solution to the \( n \)-card problem.

The length \( \frac{n}{n-1} \) in Theorem 1.2 is the smallest possible for a general interval, because the \( n \times n \) stochastic matrix having one row \( (0, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}) \) and \( n-1 \) rows \( (1,0,0,\ldots,0) \) has transversal sums of 0 and \( \frac{n}{n-1} \) only.

On the other hand, the known complete set of minimal solutions (1.2) for \( n = 3 \) and (1.3) for \( n = 4 \) shows that the interval length in Theorem 1.2 can be reduced to 1 for specific intervals. By reference to particular \( n \times n \) stochastic matrices, Sands \cite{San11} showed that every solution to the \( n \)-card problem for \( n \geq 2 \) must contain a length 1 interval \( \left[ \frac{k}{n-1}, 1+\frac{k}{n-1} \right] \) for some \( k \in \{0,1,\ldots,n-1\} \). Problem 3 of \cite{San11} asks whether each such length 1 interval is itself a solution to the \( n \)-card problem, which would imply that the complete set of minimal solutions to the \( n \)-card problem comprises these \( n \) intervals. This question remains open for \( n > 4 \).
2 The interval $[1,2]$

In this section we prove Theorem 1.1. We firstly establish some preliminary lemmas.

**Lemma 2.1.** Let $(a_{ij})$ be an $n \times n$ stochastic matrix, all of whose transversal sums lie outside an interval $[u, v]$ containing 1. Then $(a_{ij})$ has at least one transversal sum less than $u$, and at least one transversal sum greater than $v$.

**Proof.** Each entry of $(a_{ij})$ is contained in exactly $(n-1)!$ transversal sums, so the mean of all transversal sums is $(n-1)!/(\sum_{i,j} a_{ij})/n! = 1$. Therefore at least one transversal sum is at most 1, and so by assumption less than $u$. Similarly, at least one transversal sum is greater than $v$. \hfill $\square$

An immediate consequence of Lemma 2.1 is that, as noted earlier, $[0,1]$ is a solution to the $n$-card problem for all $n \geq 2$.

The rows and columns of an $n \times n$ stochastic matrix can be permuted without changing the set of its $n!$ transversal sums. Our method relies on examining the effect of transposing two rows of an $n \times n$ stochastic matrix, and thereby bounding the matrix entries. We now show that if an $n \times n$ stochastic matrix has a sufficiently large diagonal sum then there must be a transposition of two rows that decreases this diagonal sum. We prove this result for the following slightly more general case of an $n \times n$ substochastic matrix (each of whose row sums is at most 1).

**Lemma 2.2.** Let $(a_{ij})$ be an $n \times n$ substochastic matrix. Suppose $(a_{ij})$ has diagonal sum greater than 1. Then $a_{kk} + a_{\ell\ell} > a_{k\ell} + a_{\ell k}$ for some $k, \ell$.

**Proof.** Suppose, for a contradiction, that $a_{ii} + a_{jj} \leq a_{ij} + a_{ji}$ for all $i, j$. Sum this inequality over all $i, j$ to obtain $2n \sum_i a_{ii} \leq 2 \sum_{i,j} a_{ij} \leq 2n$, since by assumption the row sums of $(a_{ij})$ are each at most 1. This implies that the diagonal sum satisfies $\sum_i a_{ii} \leq 1$, giving the required contradiction. \hfill $\square$

We next give conditions under which the sum of two diagonal entries of an $n \times n$ stochastic matrix can be bounded from below.

**Lemma 2.3.** Let $(a_{ij})$ be an $n \times n$ stochastic matrix with diagonal sum $d$, and suppose all transversal sums of $(a_{ij})$ lie outside the interval $[u, d)$. Then, for all $i, j$,

$$a_{ii} + a_{jj} > a_{ij} + a_{ji} \implies a_{ii} + a_{jj} > d - u.$$ 

**Proof.** Suppose $a_{ii} + a_{jj} > a_{ij} + a_{ji}$. Then the positive quantity $a_{ii} + a_{jj} - a_{ij} - a_{ji}$ is the decrease in the diagonal sum caused by transposing rows $i$ and $j$ of the matrix, and so by assumption is greater than $d - u$. We therefore have $a_{ii} + a_{jj} \geq a_{ii} + a_{jj} - a_{ij} - a_{ji} > d - u$. \hfill $\square$

Define an $n \times n$ stochastic matrix $(a_{ij})$ to be **diagonally ordered** if its diagonal entries are in non-increasing order:

$$a_{11} \geq a_{22} \geq \cdots \geq a_{nn}.$$ 

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1. We know from (1.1) and (1.2) that the result holds for \( n = 2 \) and \( 3 \), so we may take \( n \geq 4 \). Suppose, for a contradiction, that \((a_{ij})\) is an \( n \times n \) stochastic matrix whose transversal sums all lie outside the interval \([1, 2]\). Then by Lemma 2.1, \((a_{ij})\) has a transversal sum greater than 2 and a transversal sum less than 1. Let \( 2 + \epsilon \) be the smallest transversal sum greater than 2, and reorder the rows and columns of \((a_{ij})\) so that the summands of this transversal sum occur on the matrix diagonal and so that the matrix is diagonally ordered. By Lemma 2.3 with \( d = 2 + \epsilon \) and \( u = 1 \),

\[
a_{ii} + a_{jj} > a_{ij} + a_{ji} \quad \text{implies} \quad a_{ii} + a_{jj} > 1 + \epsilon. \tag{2.1}
\]

Now the \((n - 1) \times (n - 1)\) submatrix of \((a_{ij})\) formed by deleting the first row and column has diagonal sum \(2 + \epsilon - a_{11} > 1\). Apply Lemma 2.2 to this submatrix to show that, for some distinct \( k > 1 \) and \( \ell > 1 \),

\[
a_{kk} + a_{\ell\ell} > a_{k\ell} + a_{\ell k}.
\]

We therefore have \(a_{kk} + a_{\ell\ell} > 1 + \epsilon\) by (2.1), and so

\[
a_{22} + a_{33} > 1 + \epsilon \tag{2.2}
\]

since the matrix is diagonally ordered and \( k, \ell \) are distinct. Since the diagonal sum of \((a_{ij})\) is \(2 + \epsilon\), we have

\[
a_{ii} + a_{11} \leq 2 + \epsilon - a_{22} - a_{33} \quad \text{for all} \quad i > 3,
\]

and therefore \(a_{ii} + a_{11} < 1\) for all \( i > 3 \), by (2.2). Then, since the matrix is diagonally ordered,

\[
a_{ii} + a_{jj} < 1 \quad \text{for all} \quad i, j \text{ with } i > 3,
\]

which in turn implies by (2.1) that

\[
a_{ii} + a_{jj} \leq a_{ij} + a_{ji} \quad \text{for all} \quad i, j \text{ with } i > 3. \tag{2.3}
\]

We complete the proof by showing that (2.2) and (2.3) force the sum of the entries of \((a_{ij})\) to be too large. We have

\[
\sum_{i,j} a_{ij} \geq \sum_{i \leq 3} a_{ii} + \sum_{i > 3} a_{ii} + \sum_{i > 3, j \leq 3} (a_{ij} + a_{ji}) \\
\geq (n - 2) \sum_{i \leq 3} a_{ii} + 4 \sum_{i > 3} a_{ii}
\]

by substitution from (2.3). Therefore

\[
\sum_{i,j} a_{ij} \geq (n - 2) \sum_{i \leq 3} a_{ii} + 2 \sum_{i > 3} a_{ii} \\
= (n - 4) \sum_{i \leq 3} a_{ii} + 2 \sum_{i} a_{ii} \\
\geq (n - 4)(1 + \epsilon) + 2(2 + \epsilon)
\]

by (2.2), using \( n \geq 4 \). Therefore \(\sum_{i,j} a_{ij} > n\), which is a contradiction because each row sum of \((a_{ij})\) is 1.

\[\square\]
3 Intervals of length \( \frac{n}{n-1} \)

In this section we prove Theorem 1.2.

**Proposition 3.1.** Let \( n \geq 4 \) and let \( (a_{ij}) \) be a diagonally ordered \( n \times n \) stochastic matrix. Suppose the diagonal sum \( d \) of \((a_{ij})\) satisfies \( d \in (1, 2) \). Then \( (a_{ij}) \) has a transversal sum lying in the interval \( [d - \frac{n}{n-1}, d) \).

**Proof.** Suppose, for a contradiction, that no transversal sum of \((a_{ij})\) lies in the interval \( [d - \frac{n}{n-1}, d) \). Then by Lemma 2.3 with \( u = d - \frac{n}{n-1} \),

\[
a_{ii} + a_{jj} > a_{ij} + a_{ji} \quad \text{implies} \quad a_{ii} + a_{jj} > \frac{n}{n-1}. \quad (3.1)
\]

Since \( d > 1 \), Lemma 2.2 gives

\[a_{kk} + a_{\ell\ell} > a_{k\ell} + a_{\ell k}\]

for some distinct \( k, \ell \), and it follows from (3.1) that \( a_{kk} + a_{\ell\ell} > \frac{n}{n-1} \). Since the matrix is diagonally ordered and \( k, \ell \) are distinct, this implies

\[a_{11} + a_{22} > \frac{n}{n-1} \quad (3.2)\]

and

\[a_{11} > \frac{1}{2} \cdot \frac{n}{n-1}. \quad (3.3)\]

We now claim that

\[a_{ii} + a_{jj} \leq \frac{n}{n-1} \quad \text{for all distinct } i > 1 \text{ and } j > 1. \quad (3.4)\]

Suppose otherwise, for a contradiction, so that \( a_{rr} + a_{ss} > \frac{n}{n-1} \) for some distinct \( r > 1 \) and \( s > 1 \). Since the matrix is diagonally ordered, this gives

\[a_{22} + a_{33} > \frac{n}{n-1}. \quad (3.5)\]

Therefore, for all \( i > 3 \), we have \( a_{ii} + a_{11} \leq d - a_{22} - a_{33} < \frac{n-2}{n-1} \) because \( d \leq 2 \). Since the matrix is diagonally ordered, we then have

\[a_{ii} + a_{jj} < \frac{n-2}{n-1} \quad \text{for all } i, j \text{ with } i > 3,\]

which by (3.1) implies

\[a_{ii} + a_{jj} \leq a_{ij} + a_{ji} \quad \text{for all } i, j \text{ with } i > 3. \quad (3.6)\]

Write

\[
\sum_{i,j} a_{ij} \geq \sum_{i \leq 3} a_{ii} + \sum_{i > 3, j \leq 3} (a_{ij} + a_{ji}) \\
\geq (n-2) \sum_{i \leq 3} a_{ii} + 3 \sum_{i > 3} a_{ii}
\]

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by substitution from (3.6), so that
\[ \sum_{i,j} a_{ij} \geq (n - 2) \sum_{i \leq 3} a_{ii} \]
\[ > (n - 2) \cdot \frac{3}{2} \cdot \frac{n}{n - 1} \]
from (3.3) and (3.5). Since \( \sum_{ij} a_{ij} = n \) and \( n \geq 4 \), this is a contradiction and proves the claim (3.4).

It then follows from (3.1) that
\[ a_{ii} + a_{jj} \leq a_{ij} + a_{ji} \quad \text{for all distinct } i > 1 \text{ and } j > 1. \]
Summing over all \( i, j \) satisfying \( 1 < i < j \), we find that
\[ (n - 2) \sum_{i>1} a_{ii} \leq \sum_{1<i<j} (a_{ij} + a_{ji}). \] (3.7)

Now let \( m \) be the largest integer \( i \) such that \( a_{11} + a_{ii} > \frac{n}{n-1} \). Note that \( m \geq 2 \), by (3.2). By (3.1) we have \( a_{11} + a_{ii} \leq a_{ii} + a_{i1} \) for \( i > m \), so that
\[ a_{11} \leq a_{i1} + a_{i1} \quad \text{for } i > m. \] (3.8)

We now show that (3.7) and (3.8) force the entries of \( (a_{ij}) \) to be too large. We have
\[ \sum_{i,j} a_{ij} \geq a_{11} + \sum_{i>1} a_{ii} + \sum_{i>m} (a_{1i} + a_{i1}) + \sum_{1<i<j} (a_{ij} + a_{ji}) \]
\[ \geq (n - m + 1)a_{11} + (n - 1) \sum_{i>1} a_{ii} \]
by substitution from (3.7) and (3.8). Therefore
\[ \sum_{i,j} a_{ij} \geq (n - m + 1)a_{11} + (n - 1) \sum_{1<i\leq m} a_{ii} \]
\[ > (n - m + 1)a_{11} + (n - 1)(m - 1) \left( \frac{n}{n - 1} - a_{11} \right) \]
by definition of \( m \) and the diagonal ordering of \( (a_{ij}) \), and so
\[ \sum_{i,j} a_{ij} > n(m - 1 - (m - 2)a_{11}). \]

Since \( m \geq 2 \) and \( a_{11} \leq 1 \), this implies the contradiction \( \sum_{i,j} a_{ij} > n \) and so completes the proof. \( \square \)

We now combine Proposition 3.1 with Theorem 1.1 to prove Theorem 1.2.

**Proof of Theorem 1.2.** We know from (1.1) and (1.2) that the result holds for \( n = 2 \) and \( 3 \), so we may take \( n \geq 4 \). Suppose, for a contradiction, that \( (a_{ij}) \) is an \( n \times n \) stochastic matrix whose transversal sums all lie outside the interval \( \left[ u, u + \frac{n}{n-1} \right] \) for some \( u \in \left[ 0, \frac{n-2}{n-1} \right] \). Since this interval contains 1, by Theorem 1.1 the matrix \( (a_{ij}) \) therefore has a transversal sum in the interval \( (u + \frac{n}{n-1}, 2) \). Let \( d \) be the smallest such transversal sum. Reorder the rows and columns of \( (a_{ij}) \) so that the summands of this transversal sum occur on the matrix diagonal and so that the matrix is diagonally ordered. Then by Proposition 3.1, \( (a_{ij}) \) has a transversal sum lying in the interval \( \left[ d - \frac{n}{n-1}, d \right] \). By choice of \( d \), this gives the required contradiction. \( \square \)
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References


