# Comment on revised version of "The Hadamard circulant conjecture"

Robert Craigen Jonathan Jedwab

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**Abstract**

The revised version of the claim by Hurley, Hurley and Hurley to have proved the circulant Hadamard matrix conjecture is mistaken.

In January 2011, Hurley, Hurley and Hurley [2] claimed to have proved the circulant Hadamard matrix conjecture, but the proof was mistaken [1]. In September 2011, a revised version [3] of the paper [2] was posted to the arXiv, with the comment that "This is post publication revision of on-line Bull. London Math. Soc. version which changes subsection 3.3." We show that the revised version is also mistaken, by summarising part of the argument of [3] and then presenting a counterexample.

A *2-block* is a matrix of the form $D = \begin{bmatrix} i & j \\ j & i \end{bmatrix}$ for $i, j \in \{1, -1\}$, and is *even* if $i = j$ and *odd* if $i = -j$. Suppose there exists a circulant Hadamard matrix $H$ of order $4n$. Reorder the rows and columns of $H$ to form a $2n \times 2n$ matrix $M$ whose entries are 2-blocks, as in [3, p.7], and write the first row of $M$ as $\begin{bmatrix} M_0 & M_1 & \dots & M_{2n-1} \end{bmatrix}$. Then exactly $n$ of the 2-blocks $M_i$ are even, and

$$\sum_{i \,:\, M_i \text{ and } M_{i+u} \text{ are even}} M_i M_{i+u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for each } u \neq 0, \tag{1}$$

where all matrix subscripts are reduced modulo $2n$. Fix $u \neq 0$. Then from (1), for each $i$ such that $M_i$ and $M_{i+u}$ are even, we can assign a unique $\ell$ such that $M_\ell$ and $M_{\ell+u}$ are even and such that $M_\ell M_{\ell+u} = -M_i M_{i+u}$. We then also assign $i$ to $\ell$, write $(i, i+u) \sim (\ell, \ell+u)$, and call the index pairs $(i, i+u)$ and $(\ell, \ell+u)$ *matching*.

An even 2-block $M_i$ is *symmetric* when the 2-block $M_{i+n}$ is also even. The following argument is given [3, p.8] to claim that "every even block is symmetric" when $n > 1$. Suppose, for a contradiction, that $M_i$ is an even block that is not symmetric. Since $n > 1$, there is an even 2-block $M_{i+u}$ for some $u \neq 0$, and there must be a pair matching $(i, i+u)$. In each of five exhaustive cases,

R. Craigen is with Department of Mathematics, University of Manitoba, Winnipeg MB R3T 2N2, Canada.

J. Jedwab is with Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby BC V5A 1S6, Canada.

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Email: craigenr@cc.umanitoba.ca, jed@sfu.ca

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this forces the existence of a further pair of even 2-blocks $(M_j, M_{j+v})$ for some $j$ and $v$, where $M_j$ is not symmetric, and there must be a pair matching $(j, j+v)$. Repeat this procedure. Since this procedure "cannot continue indefinitely," we obtain a contradiction.

The following is a counterexample to this claimed procedure, using $n = 3$ and only the first of the five specified cases:

$$(M_0, M_1, M_2, M_3, M_4, M_5) = \left( \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} - & - \\ - & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix} \right)$$

(writing $+$ for 1 and $-$ for $-1$). The even 2-blocks are $M_0$, $M_2$, and $M_4$, none of which is symmetric. Assign the matchings $(0,2) \sim (2,4)$ and $(0,4) \sim (4,2)$. Let $i = 0$ and $j = 2$, and follow the procedure of [3, p.8]. Since $(0,2) \sim (2,4)$, there must be a pair matching $(0,4)$. Then, since $(0,4) \sim (4,2)$, there must be a pair matching $(0,2)$. However $(0,2)$ already has a matching pair $(2,4)$, so the claimed contradiction does not arise.

## References


[1] R. Craigen and J. Jedwab. Comment on "The Hadamard circulant conjecture". `arXiv:1111.3437v1 [math.CO]`.

[2] B. Hurley, P. Hurley, and T. Hurley. The Hadamard circulant conjecture. *Bull. London Math. Soc.*, 2011. doi:10.1112/blms/bdq112.

[3] B. Hurley, P. Hurley, and T. Hurley. The Hadamard circulant conjecture. `arXiv:1109.0748v1 [math.RA]`.