Construction and nonexistence of strong external difference families

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Abstract

Strong external difference families (SEDFs) were introduced by Paterson and Stinson as a more restrictive version of external difference families. SEDFs can be used to produce optimal strong algebraic manipulation detection codes. We characterize the parameters (v, m, k, λ) of a nontrivial SEDF that is near-complete (satisfying $v = km+1$). We construct the first known nontrivial example of a (v, m, k, λ) SEDF having $m > 2$. The parameters of this example are $(243, 11, 22, 20)$, giving a near-complete SEDF, and its group is $\mathbb{Z}_3^5$. We provide a comprehensive framework for the study of SEDFs using character theory and algebraic number theory, showing that the cases $m = 2$ and $m > 2$ are fundamentally different. We prove a range of nonexistence results, greatly narrowing the scope of possible parameters of SEDFs.

Keywords. Construction, exponent bound, near-complete, nonexistence, strong external difference family.

1 Introduction

Let G be an abelian group of order v with identity 1. We shall work in the setting of the group ring $\mathbb{Z}[G]$: given a subset D of G, we write the group ring element $\sum_{d\in D} d$ as D (by a standard abuse of notation), and the group ring element $\sum_{d\in D} d^{-1}$ as $D^{(-1)}$. Let $D_1, D_2, \ldots, D_m$ be mutually disjoint k-subsets of G, where $m \geq 2$, and let λ be a positive integer. Then $\{D_1, D_2, \ldots, D_m\}$ is a (v, m, k, λ)-*external difference family* in G if

$$\sum_{\substack{1\leq i,j\leq m\\ i\neq j}} D_j D_i^{(-1)} = \lambda(G-1) \quad \text{in } \mathbb{Z}[G], \tag{1.1}$$

and is a (v, m, k, λ)-*strong external difference family* (SEDF) in G if

$$D_j \sum_{\substack{1\leq i\leq m\\ i\neq j}} D_i^{(-1)} = \lambda(G-1) \quad \text{in } \mathbb{Z}[G] \text{ for each } j \text{ satisfying } 1 \leq j \leq m. \tag{1.2}$$

The use of "strong" arises because a (v, m, k, λ)-SEDF is necessarily a $(v, m, k, m\lambda)$-external difference family.

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External difference families have applications in authentication codes and secret sharing [22]. An external difference family in a cyclic group gives rise to difference systems of sets [7], which can be applied to construct synchronization codes [18]. Paterson and Stinson [24] introduced SEDFs and showed how to produce optimal strong algebraic manipulation detection codes from them. Algebraic manipulation detection codes have many applications, including robust secret sharing schemes, secure multiparty computation, and non-malleable codes [8, 9, 10]. A succession of recent papers has demonstrated that SEDFs are interesting combinatorial objects in their own right: see Proposition 1.1 below for a summary of constructive results, Proposition 1.2 for a characterization result, and Proposition 1.3 for a selection of nonexistence results.

The parameters of a (v, m, k, λ)-SEDF satisfy the counting relation

$$k^2(m-1) = \lambda(v-1). \tag{1.3}$$

A (v, m, k, λ)-SEDF is *trivial* if $k = 1$; it follows from (1.3) that the parameters of a trivial SEDF have the form $(v, v, 1, 1)$, and an SEDF with these parameters exists (trivially) in every group of order v. The following proposition describes the parameters and groups of the known nontrivial SEDFs, all of which satisfy $m = 2$.

Proposition 1.1. *A (v, m, k, λ)-SEDF exists in the group G in each of the following cases:*

(1) $(v, m, k, \lambda) = (k^2 + 1, 2, k, 1)$ *and* $G = \mathbb{Z}_{k^2+1}$ [24, Example 2.2]

(2) $(v, m, k, \lambda) = (v, 2, \frac{v-1}{2}, \frac{v-1}{4})$ *and* $v \equiv 1 \pmod 4$*, provided there exists a* $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$ *partial difference set in* G [11, Section 3], [15, Theorem 4.4]

(3) $(v, m, k, \lambda) = (q, 2, \frac{q-1}{4}, \frac{q-1}{16})$ *where* $q = 16t^2 + 1$ *is a prime power and* t *is an integer, and* $G = \mathbb{F}_q$ [3, Theorem 4.3]

(4) $(v, m, k, \lambda) = (q, 2, \frac{q-1}{6}, \frac{q-1}{36})$ *where* $q = 108t^2 + 1$ *is a prime power and* t *is an integer, and* $G = \mathbb{F}_q$ [3, Theorem 4.6].

When $\lambda = 1$, the parameters of a nontrivial (v, m, k, λ)-SEDF have been characterized.

Proposition 1.2. *A nontrivial $(v, m, k, 1)$-SEDF exists if and only if $m = 2$ and $v = k^2 + 1$* [24, Theorem 2.3].

The following proposition describes parameter sets (v, k, m, λ) for which a nontrivial SEDF is known not to exist in all groups of order v.

Proposition 1.3. *A nontrivial (v, m, k, λ)-SEDF does not exist in each of the following cases:*

(1) $m \in \{3, 4\}$ [21, Theorems 3.3 and 3.6]

(2) $m > 2$ *and v is prime* [21, Theorem 3.9]

(3) $m > 2$ *and* $\lambda = 2$ [15, Corollary 3.2]

(4) $m > 2$ *and* $\lambda > 1$ *and* $\frac{\lambda(k-1)(m-2)}{(\lambda-1)k(m-1)} > 1$ [15, Theorem 3.5]

(5) $m > 2$ *and there is a prime p dividing v for which* $\gcd(km, p) = 1$ *and* $m \not\equiv 2 \pmod p$ [3, Theorem 3.5].

(6) $\lambda \geq k$ [3, Lemma 1.1]

It is known [21, Lemma 1.2] that if $v = km$, then an (v, m, k, λ)-SEDF is necessarily trivial. The same proof idea as in [21] gives the following generalization.

Lemma 1.4. *Suppose there exists a* (v, m, k, λ)*-SEDF for which* $\gcd(k, v-1) = 1$*. Then the SEDF is trivial.*

Proof. The counting relation (1.3) gives $m - 1 = \frac{\lambda}{k^2}(v-1)$. Since $\gcd(k, v-1) = 1$, it follows that λ/k^2 is an integer and so $m - 1 \geq v - 1$. Since $v \geq km$, this implies that $k = 1$. □

Lemma 1.4 implies that the parameters of a nontrivial (v, m, k, λ)-SEDF $\{D_1, D_2, \ldots, D_m\}$ satisfy $v > km$ and, by taking a translate of all the subsets D_j if necessary, we may therefore assume that $1 \notin \bigcup_{j=1}^{m} D_j$. In the extremal case $v = km+1$, the subsets $D_1, D_2, \ldots, D_m$ partition the nonidentity elements of the group G and (following [11]) we call the SEDF *near-complete*.

In this paper, we present constructive and nonexistence results for nontrivial SEDFs using character theory and algebraic number theory. In Section 2, we give a character-theoretic framework for the study of SEDFs and demonstrate that the cases $m = 2$ and $m > 2$ are fundamentally different. In Section 3, we characterize the parameters of a nontrivial near-complete SEDF by establishing an equivalence with a collection of partial difference sets. In particular, we construct a near-complete $(243, 11, 22, 20)$-SEDF in $\mathbb{Z}_3^5$ by reference to the point-orbits of the Mathieu group M_{11} acting on the projective geometry $PG(4, 3)$. This is the first known nontrivial example of an SEDF with $m > 2$. In Section 4, we use algebraic number theory to obtain an exponent bound on a group containing a SEDF and apply it to rule out various SEDFs with $m = 2$, leaving only 5 open cases for the parameters of a (v, m, k, λ) SEDF with $v \leq 50$ and $m = 2$. In Section 5 we obtain nonexistence results for SEDFs with $m > 2$, introducing the "simple character value property" under which strong necessary conditions can be derived. This leaves only 70 open cases for the parameters of a (v, m, k, λ) SEDF with $v \leq 10^4$ and $m > 2$.

2 A character-theoretic approach

Let $\widehat{G}$ denote the character group of an abelian group G, and let $\chi_0 \in \widehat{G}$ be the principal character. Each character $\chi \in \widehat{G}$ is extended linearly to the group ring $\mathbb{Z}[G]$. The following formula is a consequence of the orthogonality properties of characters.

Proposition 2.1 (Fourier inversion formula)**.** *Let* G *be an abelian group and let* $A = \sum_{g \in G} c_g g \in \mathbb{Z}[G]$*. Then*

$$c_g = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \chi(A)\overline{\chi(g)} \quad \textit{for each } g \in G.$$

Suppose $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, k, m, λ)-SEDF in a group G, and write $D = \bigcup_{i=1}^{m} D_i$. Then (1.2) is equivalent to

$$D_j(D^{(-1)} - D_j^{(-1)}) = \lambda(G - 1) \quad \text{in } \mathbb{Z}[G] \text{ for each } j \text{ satisfying } 1 \leq j \leq m.$$

Apply a nonprincipal character $\chi \in \widehat{G}$ to obtain

$$\chi(D_j)\overline{(\chi(D) - \chi(D_j))} = -\lambda \quad \text{for all nonprincipal } \chi \in \widehat{G} \text{ and for each } j. \tag{2.1}$$

Some basic restrictions were derived from (2.1) in [21, Lemma 3.1]. We now extend that analysis.

It follows from (2.1) that for each j satisfying $1 \leq j \leq m$,

$$|\chi(D_j)|^2 = \lambda \quad \text{if and only if} \quad \chi(D) = 0. \tag{2.2}$$

Define

$$\widehat{G}^0 = \{\text{nonprincipal } \chi \in \widehat{G} \mid \chi(D) = 0\}, \tag{2.3}$$
$$\widehat{G}^N = \{\text{nonprincipal } \chi \in \widehat{G} \mid \chi(D) \neq 0\},$$

so that $\widehat{G}$ may be partitioned (with respect to D) as the disjoint union $\{\chi_0\} \cup \widehat{G}^0 \cup \widehat{G}^N$. We now show that the set $\widehat{G}^N$ is non-empty.

Lemma 2.2 ([21, Lemma 3.1 (d)]). *Suppose $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, k, m, λ)-SEDF in a group G, and let $D = \bigcup_{i=1}^m D_i$. Then $|\widehat{G}^N| > 0$.*

Proof. Suppose, for a contradiction, that $\chi(D) = 0$ for each nonprincipal $\chi \in \widehat{G}$. Write $D = \sum_{g \in G} c_g g$ in $\mathbb{Z}[G]$ and use Proposition 2.1 to show that for each $g \in G$ we have

$$c_g = \frac{1}{v}\chi_0(D)\overline{\chi_0(g)} = \frac{km}{v}.$$

By Lemma 1.4 we have $v > km$, giving the contradiction $0 < c_g < 1$. □

For each $\chi \in \widehat{G}^N$, set $\alpha_{j,\chi}$ to be the real number $\frac{|\chi(D_j)|^2}{|\chi(D_j)|^2 - \lambda}$. Then conjugate (2.1), multiply both sides by $\chi(D_j)$, and rearrange to give

$$\chi(D_j) = \alpha_{j,\chi}\,\chi(D) \quad \text{for } \chi \in \widehat{G}^N. \tag{2.4}$$

Substitute for $\chi(D_j)$ from (2.4) into (2.1) to obtain a quadratic equation in $\alpha_{j,\chi}$:

$$\alpha_{j,\chi}^2 - \alpha_{j,\chi} - \tfrac{\lambda}{|\chi(D)|^2} = 0 \quad \text{for } \chi \in \widehat{G}^N. \tag{2.5}$$

The solutions of this equation are

$$\alpha_\chi^+ = \frac{1}{2}\Big(1 + \sqrt{1 + \tfrac{4\lambda}{|\chi(D)|^2}}\Big), \quad \alpha_\chi^- = \frac{1}{2}\Big(1 - \sqrt{1 + \tfrac{4\lambda}{|\chi(D)|^2}}\Big) \quad \text{for } \chi \in \widehat{G}^N. \tag{2.6}$$

For each $\chi \in \widehat{G}^N$, let ℓ_χ^+ and ℓ_χ^- be the number of times $\alpha_{j,\chi}$ takes the value α_χ^+ and α_χ^-, respectively, as j ranges over $1 \le j \le m$. Using $\chi(D) = \sum_{j=1}^m \chi(D_j)$, we find from (2.4) that

$$\ell_\chi^+ \alpha_\chi^+ + \ell_\chi^- \alpha_\chi^- = 1.$$

Combine with the counting condition $\ell_\chi^+ + \ell_\chi^- = m$ to determine ℓ_χ^+ and ℓ_χ^- as

$$\ell_\chi^+ = \frac{m}{2} - \frac{m-2}{2\sqrt{1 + \frac{4\lambda}{|\chi(D)|^2}}}, \quad \ell_\chi^- = \frac{m}{2} + \frac{m-2}{2\sqrt{1 + \frac{4\lambda}{|\chi(D)|^2}}} \qquad \text{for } \chi \in \widehat{G}^N. \tag{2.7}$$

In particular, $\ell_\chi^+ \ge \frac{m}{2} - \frac{m-2}{2} = 1$ and $\ell_\chi^- \ge 1$, so the values α_χ^+ and α_χ^- both occur as j ranges over $\{1, 2, \ldots, m\}$. Therefore from (2.4) we have

$$\{\chi(D_j) \mid 1 \le j \le m\} = \{\alpha_\chi^+\,\chi(D), \alpha_\chi^-\,\chi(D)\} \quad \text{for } \chi \in \widehat{G}^N. \tag{2.8}$$

The expressions (2.7) illustrate a fundamental difference between the cases $m = 2$ and $m > 2$. When $m = 2$, these expressions reduce to $\ell_\chi^+ = \ell_\chi^- = 1$. But when $m > 2$, we require $\sqrt{1 + \frac{4\lambda}{|\chi(D)|^2}} \in \mathbb{Q}$ for each $\chi \in \widehat{G}^N$ in order for ℓ_χ^+ and ℓ_χ^- to be integers. We shall see in Section 5 that this yields strong restrictions on the character values of $\chi(D)$ and $\chi(D_j)$ for SEDFs when $m > 2$, which do not apply when $m = 2$.

We conclude this section with a result required in Section 3.

Lemma 2.3. *Suppose $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, m, k, λ)-SEDF in a group G, where $1 \notin \bigcup_{i=1}^{m} D_i$. Then, for each j, neither $D_j \cup \{1\}$ nor $G \setminus D_j$ is a subgroup of G.*

Proof. Suppose, for a contradiction, that $D_j \cup \{1\}$ is a subgroup of G. Since $m \geq 2$, there exists a nonprincipal character χ of G which is principal on $D_j \cup \{1\}$. Then (2.1) gives $k(\overline{\chi(D)} - k) = -\lambda$, so that $\chi(D) = k - \frac{\lambda}{k}$ is a rational number. Since $\chi(D)$ is also an algebraic integer, λ/k is an integer and therefore $\lambda \geq k$. This contradicts Proposition 1.3 (6).

Suppose, for a contradiction, that $G \setminus D_j$ is a subgroup of G. Then $(v - k) \mid v$, and since $k > 1$ we have $k \geq \frac{v}{2}$. But $v > km$ and $m \geq 2$ gives the contradiction $k < \frac{v}{2}$. □

3 Near-complete SEDFs

Let D be a k-subset of a group G of order v, where $1 \notin D$. The subset D is a (v, k, λ, μ) *partial difference set* (PDS) in G if

$$DD^{(-1)} = (k - \mu) \cdot 1 + \lambda D + \mu(G - D) \quad \text{in } \mathbb{Z}[G]. \tag{3.1}$$

(A slightly different definition, which we will not require, applies when $1 \in D$.) The PDS D is *regular* if $D = D^{(-1)}$, and is *trivial* if either $D \cup \{1\}$ or $G \setminus D$ is a subgroup of G. In this section we prove the following result, which characterizes the parameters of a nontrivial near-complete (v, m, k, λ)-SEDF and provides the first known example of a nontrivial (v, m, k, λ)-SEDF having $m > 2$.

Theorem 3.1. *Let $D_1, D_2, \ldots, D_m$ partition the nonidentity elements of an abelian group G of order $v = km + 1$ into m subsets each of size $k > 1$. Then $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial near-complete (v, m, k, λ)-SEDF in G if and only if either*

(1) $(v, m, k, \lambda) = (v, 2, \frac{v-1}{2}, \frac{v-1}{4})$ and $v \equiv 1 \pmod 4$ and D_1 is a nontrivial regular $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$-PDS in G, or

(2) $(v, m, k, \lambda) = (243, 11, 22, 20)$ and each D_j is a nontrivial regular $(243, 22, 1, 2)$-PDS in G for $1 \leq j \leq 11$.

Furthermore, a $(243, 11, 22, 20)$-SEDF exists in $\mathbb{Z}_3^5$.

The restriction of Theorem 3.1 to the case $m = 2$ is due to Huczynska and Paterson [15, Theorem 4.6], and also to Ding [12, Proposition 2.1] from the viewpoint of difference families. One direction of the case $m = 2$, namely the construction of an SEDF from a PDS, was also proved in [11, Section 3]. Necessary and sufficient conditions for the existence of a PDS with the parameters specified in (1) and (2) of Theorem 3.1 are not known. However, sufficient conditions for the existence of a PDS with the parameters specified in (1) of Theorem 3.1 (known as a Paley-type PDS) are known to include: G is elementary abelian and v is a prime power congruent to 1 modulo 4 [23]; $G = \mathbb{Z}_{p^r}^2$ for an odd prime p [17]; and $G = \mathbb{Z}_3^2 \times \mathbb{Z}_p^{4r}$ for an odd prime p [25]. Necessary conditions for the existence of a PDS in an abelian group G with the parameters specified in (2) of Theorem 3.1 are that $G = \mathbb{Z}_3^5$, $\mathbb{Z}_3^3 \times \mathbb{Z}_9$, or $\mathbb{Z}_3 \times \mathbb{Z}_9^2$ [20, Theorem 6.9]; existence is known for $G = \mathbb{Z}_3^5$ [4], [6, Section 10].

In order to establish Theorem 3.1, we make the following connection between a nontrivial near-complete SEDF and a collection of nontrivial regular PDSs.

Lemma 3.2. *Let $D_1, D_2, \ldots, D_m$ partition the nonidentity elements of an abelian group G of order $v = km+1$ into m subsets each of size $k > 1$. Then $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial near-complete (v, m, k, λ)-SEDF in G if and only if each D_j is a nontrivial regular $(v, k, k-\lambda-1, k-\lambda)$-PDS in G for $1 \leq j \leq m$.*

Proof. Since $D_1, D_2, \ldots, D_m$ is a partition of the nonidentity elements of G, for each j satisfying $1 \le j \le m$ we have $1 \notin D_j$ and

$$\sum_{\substack{1 \le i \le m \\ i \ne j}} D_i = G - D_j - 1,$$

and therefore

$$D_j \sum_{\substack{1 \le i \le m \\ i \ne j}} D_i^{(-1)} = D_j(G - D_j^{(-1)} - 1) = kG - D_j D_j^{(-1)} - D_j.$$

It follows that $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial near-complete (v, m, k, λ)-SEDF in G if and only if, for each j,

$$\lambda(G-1) = kG - D_j D_j^{(-1)} - D_j,$$

which rearranges to

$$D_j D_j^{(-1)} = \lambda \cdot 1 + (k - \lambda - 1)D_j + (k - \lambda)(G - D_j). \tag{3.2}$$

Equivalently, each D_j is a $(v, k, k-\lambda-1, k-\lambda)$-PDS in G.

To complete the proof, we require that if $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial near-complete (v, m, k, λ)-SEDF in G, then each PDS D_j is nontrivial and regular. Nontriviality of each D_j is given by Lemma 2.3, and regularity by [20, Proposition 1.2]. □

The parameters of the nontrivial regular PDSs specified in Lemma 3.2 take the form $(v, k, \mu - 1, \mu)$. The following result characterizes all such parameters when the group is abelian.

Theorem 3.3 ([2]; see also [20, Theorem 13.1])**.** *Suppose there exists a nontrivial regular $(v, k, \mu-1, \mu)$-PDS in an abelian group. Then either*

(1) $(v, k, \mu-1, \mu) = (v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$ *and* $v \equiv 1 \pmod 4$*, or*

(2) $(v, k, \mu-1, \mu) = (243, 22, 1, 2)$ *or* $(243, 220, 199, 220)$*.*

We can now give the structure of the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial near-complete (v, m, k, λ)-SEDF in G if and only if each D_j is a nontrivial regular $(v, k, k-\lambda-1, k-\lambda)$-PDS in G for $1 \le j \le m$. Since $m = (v-1)/k$, by Theorem 3.3 this holds if and only if either

(1) $(v, m, k, \lambda) = (v, 2, \frac{v-1}{2}, \frac{v-1}{4})$ and $v \equiv 1 \pmod 4$ and each of D_1, D_2 is a nontrivial regular $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$-PDS in G, or

(2) $(v, m, k, \lambda) = (243, 11, 22, 20)$ and each D_j is a nontrivial regular $(243, 22, 1, 2)$-PDS in G for $1 \le j \le 11$.

For case (1), the desired result follows from the observation that if D_1 is a regular $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$-PDS in G that does not contain the identity, then so is $D_2 = G \setminus (D \cup \{1\})$ [15, Lemma 4.3].

It remains to construct a $(243, 11, 22, 20)$-SEDF in $\mathbb{Z}_3^5$, which is carried out below. □

In the rest of this section we shall construct a $(243, 11, 22, 20)$-SEDF in $\mathbb{Z}_3^5$, regarded as the additive group of $\mathbb{F}_3^5$. By Lemma 3.2, this is equivalent to partitioning the nonzero elements of $\mathbb{F}_3^5$ into 11 subsets, each of which is a nontrivial regular $(243, 22, 1, 2)$-PDS in the additive group of $\mathbb{F}_3^5$.

We firstly review the construction of a single nontrivial regular $(243, 22, 1, 2)$-PDS in the additive group of $\mathbb{F}_3^5$. This PDS was originally constructed from the perfect ternary Golay code [4]; we shall use the following alternative description involving a group of collineations of projective space having exactly two point-orbits [6, Section 10]. The $\frac{3^5-1}{3-1} = 121$ points of the projective space $\mathrm{PG}(4,3)$ are the 1-dimensional subspaces of the vector space $\mathbb{F}_3^5$ over $\mathbb{F}_3$. Each such point has the form $\langle x \rangle$ for some nonzero $x \in \mathbb{F}_3^5$, and corresponds to the vectors x and $2x$ of $\mathbb{F}_3^5$. The general linear group $\mathrm{GL}(5,3)$ is the group of 5×5 invertible matrices over $\mathbb{F}_3$, and its center is $Z = \{I, 2I\}$ where I is the 5×5 identity matrix. The projective linear group $\mathrm{PGL}(5,3)$ is the quotient group $\mathrm{GL}(5,3)/Z$. The action of an element $A \in \mathrm{PGL}(5,3)$ on a point $\langle x \rangle \in \mathrm{PG}(4,3)$ is given by

$$A : \langle x \rangle \mapsto \langle xA \rangle,$$

where xA is the usual vector-matrix product, and this action is transitive on the points of $\mathrm{PG}(4,3)$ [13, p. 57]. Now $\mathrm{PGL}(5,3)$ contains a subgroup of order 7920 which is a representation of the Mathieu group M_{11}. The group M_{11} has exactly two point-orbits on $\mathrm{PG}(4,3)$: one of size 11 and the other of size 110 [6, Example RT6]. The 22 vectors of $\mathbb{F}_3^5$ corresponding to the point-orbit of size 11 form a nontrivial regular $(243, 22, 1, 2)$-PDS in the additive group of $\mathbb{F}_3^5$ [6, Theorem 3.2 and Figure 2b].

Define the elements of $\mathrm{PGL}(5,3)$:

$$X = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 & 1 \end{bmatrix},$$

which satisfy $X^2 = Y^4 = (XY)^{11} = I$. The group M_{11} may be represented explicitly [1] as

$$M_{11} = \langle X, Y \rangle.$$

The software package *Magma* gives the point-orbit of size 11 under the action of M_{11} on $\mathrm{PG}(4,3)$ as

$$\begin{aligned} O_1 = \{&\langle(1,0,0,0,0)\rangle, \langle(1,1,0,0,2)\rangle, \langle(2,2,1,0,1)\rangle, \langle(1,0,2,1,0)\rangle, \langle(0,0,2,1,2)\rangle, \langle(0,1,2,0,0)\rangle, \\ &\langle(0,0,1,0,1)\rangle, \langle(2,0,0,2,1)\rangle, \langle(2,2,1,2,0)\rangle, \langle(0,1,0,1,2)\rangle, \langle(0,2,0,2,0)\rangle\}, \end{aligned} \tag{3.3}$$

and the corresponding nontrivial regular $(243, 22, 1, 2)$-PDS is

$$\begin{aligned} B_1 &= \{x \mid \langle x \rangle \in O_1\} \cup \{2x \mid \langle x \rangle \in O_1\} \\ &= \{(1,0,0,0,0), (1,1,0,0,2), (2,2,1,0,1), (1,0,2,1,0), (0,0,2,1,2), (0,1,2,0,0), \\ &\quad (0,0,1,0,1), (2,0,0,2,1), (2,2,1,2,0), (0,1,0,1,2), (0,2,0,2,0), (2,0,0,0,0), \\ &\quad (2,2,0,0,1), (1,1,2,0,2), (2,0,1,2,0), (0,0,1,2,1), (0,2,1,0,0), (0,0,2,0,2), \\ &\quad (1,0,0,1,2), (1,1,2,1,0), (0,2,0,2,1), (0,1,0,1,0)\} \end{aligned} \tag{3.4}$$

in the additive group of $\mathbb{F}_3^5$.

It is convenient to write

$$W = XY = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{bmatrix},$$

giving the alternative representation

$$M_{11} = \langle W, Y \rangle.$$

Now the cyclic group $\langle W \rangle$ is an order 11 subgroup of M_{11} and so fixes the set O_1. The orbit of $\langle (1,0,0,0,0) \rangle$ under the action of $\langle W \rangle$ has size 1 or 11; since $\langle W \rangle$ does not fix the point $\langle (1,0,0,0,0) \rangle$, this orbit is the whole of O_1:

$$O_1 = \{ \langle (1,0,0,0,0)A \rangle \mid A \in \langle W \rangle \}. \tag{3.5}$$

Recall that the group M_{11} has exactly two point-orbits on PG$(4,3)$: one of size 11 (the set O_1), and the other of size 110. We will show that the action of the cyclic subgroup $\langle W \rangle$ of M_{11} on the points of PG$(4,3)$ breaks the point-orbit of size 110 (under the action of M_{11}) into 10 point-orbits of size 11, each of which also corresponds to a nontrivial regular $(243, 22, 1, 2)$-PDS in the additive group of $\mathbb{F}_3^5$. This will give the partition of the nonzero elements of $\mathbb{F}_3^5$ into 11 subsets required under Lemma 3.2.

The centralizer of W in PGL$(5,3)$ is the group $C(W) = \{B \in \mathrm{PGL}(5,3) : BW = WB\}$. *Magma* gives $C(W)$ to be a cyclic group of order 121, one of whose generators is

$$S = \begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix},$$

which satisfies $W = S^{11}$. Define subsets $O_2, O_3, \ldots, O_{11}$ of PG$(4,3)$ by

$$O_j = \{ \langle x S^{j-1} \rangle \mid \langle x \rangle \in O_1 \} \quad \text{for } 2 \le j \le 11. \tag{3.6}$$

Then for $1 \le j \le 11$ we find from (3.5) that

$$\begin{aligned} O_j &= \{ \langle (1,0,0,0,0) A S^{j-1} \rangle \mid A \in \langle W \rangle \} \\ &= \{ \langle (1,0,0,0,0) S^{j-1} A \rangle \mid A \in \langle W \rangle \} \end{aligned}$$

because $S \in C(W)$, and therefore the subset O_j is the size 11 orbit of the point $\langle (1,0,0,0,0) S^{j-1} \rangle$ under the action of $\langle W \rangle$. Furthermore, using $W = S^{11}$ we may write

$$O_j = \{ \langle (1,0,0,0,0) S^{11i+j-1} \rangle \mid 0 \le i \le 10 \} \quad \text{for } 1 \le j \le 11, \tag{3.7}$$

so that

$$\bigcup_{j=1}^{11} O_j = \{ \langle (1,0,0,0,0) S^{\ell} \rangle \mid 0 \le \ell \le 120 \}. \tag{3.8}$$

We claim that the subsets $O_1, O_2, \ldots, O_{11}$ form a partition of the 121 points of PG$(4,3)$. Suppose, for a contradiction, that there is an integer n satisfying $1 \le n \le 120$ such that

$$\langle (1,0,0,0,0) S^n \rangle = \langle (1,0,0,0,0) \rangle. \tag{3.9}$$

Since $\langle S \rangle = C(W)$ has order 121, the matrix S^n has order 11 or 121. But S^n cannnot have order 121, otherwise S would fix the point $\langle (1,0,0,0,0) \rangle$ and then from (3.8) we would have $\bigcup_{j=1}^{11} O_j = \{ \langle (1,0,0,0,0) \rangle \}$, contradicting (3.3). Therefore S^n has order 11, so $S^n = S^{11i}$ for some i satisfying $1 \le i \le 10$. But from (3.7) the 11 points $\{ \langle (1,0,0,0,0) S^{11i} \rangle \mid 0 \le i \le 10 \}$

comprise the orbit O_1, and from (3.3) these 11 points are all distinct. This contradicts (3.9) and establishes the claim.

Finally, define subsets $B_2, B_3, \ldots, B_{11}$ of the nonzero elements of $\mathbb{F}_3^5$ by setting

$$B_j = \{x \mid \langle x \rangle \in O_j\} \cup \{2x \mid \langle x \rangle \in O_j\} \quad \text{for } 2 \le j \le 11.$$

The subsets $B_1, B_2, \ldots, B_{11}$ partition the 242 nonzero elements of $\mathbb{F}_3^5$, and from (3.4) and (3.6) we have

$$B_j = \{xS^{j-1} \mid x \in B_1\} \quad \text{for } 1 \le j \le 11. \tag{3.10}$$

Moreover, B_1 is a nontrivial regular $(243, 22, 1, 2)$-PDS in the additive group of $\mathbb{F}_3^5$, so from the definition (3.1) the multiset $\{x - y \mid x, y \in B_1\}$ contains the element 0 exactly 22 times, each element of B_1 exactly once, and each other element of $\mathbb{F}_3^5$ exactly twice. Since S is invertible, it follows from (3.10) that each B_j is also a nontrivial regular $(243, 22, 1, 2)$-PDS in the additive group of $\mathbb{F}_3^5$ for $2 \le j \le 11$. By Lemma 3.2, $\{B_1, B_2, \ldots, B_{11}\}$ is therefore a $(243, 11, 22, 20)$ near-complete SEDF in the additive group of $\mathbb{F}_3^5$.

Explicitly, we have

$$\begin{aligned}
B_2 = \{&(1,2,2,1,2),(1,0,1,2,0),(2,2,1,2,2),(1,0,0,0,2),(2,0,0,2,2),(1,0,1,0,0),\\
&(1,0,0,1,0),(0,2,2,2,0),(1,1,0,2,2),(0,1,2,1,0),(2,1,0,2,2),(2,1,1,2,1),\\
&(2,0,2,1,0),(1,1,2,1,1),(2,0,0,0,1),(1,0,0,1,1),(2,0,2,0,0),(2,0,0,2,0),\\
&(0,1,1,1,0),(2,2,0,1,1),(0,2,1,2,0),(1,2,0,1,1)\},\\
B_3 = \{&(2,0,2,2,2),(1,1,1,2,2),(0,0,2,2,1),(0,1,1,1,1),(1,0,0,2,1),(1,1,1,2,1),\\
&(1,2,2,1,1),(2,2,1,1,0),(1,0,1,2,1),(1,0,1,0,2),(2,2,0,0,0),(1,0,1,1,1),\\
&(2,2,2,1,1),(0,0,1,1,2),(0,2,2,2,2),(2,0,0,1,2),(2,2,2,1,2),(2,1,1,2,2),\\
&(1,1,2,2,0),(2,0,2,1,2),(2,0,2,0,1),(1,1,0,0,0)\},\\
B_4 = \{&(1,1,1,1,2),(1,2,0,0,0),(1,2,2,2,0),(2,2,0,2,1),(2,0,0,1,1),(0,1,2,0,2),\\
&(1,2,1,2,1),(1,1,0,2,0),(2,2,2,2,0),(0,0,0,2,0),(1,2,1,1,2),(2,2,2,2,1),\\
&(2,1,0,0,0),(2,1,1,1,0),(1,1,0,1,2),(1,0,0,2,2),(0,2,1,0,1),(2,1,2,1,2),\\
&(2,2,0,1,0),(1,1,1,1,0),(0,0,0,1,0),(2,1,2,2,1)\},\\
B_5 = \{&(1,2,0,0,1),(0,0,2,0,0),(0,1,0,2,2),(2,0,2,1,1),(0,2,2,2,1),(0,2,0,0,2),\\
&(1,0,2,1,1),(2,1,2,2,2),(1,0,2,0,1),(0,0,0,0,1),(2,1,0,1,0),(2,1,0,0,2),\\
&(0,0,1,0,0),(0,2,0,1,1),(1,0,1,2,2),(0,1,1,1,2),(0,1,0,0,1),(2,0,1,2,2),\\
&(1,2,1,1,1),(2,0,1,0,2),(0,0,0,0,2),(1,2,0,2,0)\},\\
B_6 = \{&(1,1,0,0,1),(0,1,1,2,1),(0,1,2,1,2),(0,0,0,1,2),(0,0,2,1,1),(1,0,2,2,0),\\
&(2,1,1,0,0),(2,0,1,2,1),(2,1,1,0,1),(1,1,1,0,1),(0,0,1,0,2),(2,2,0,0,2),\\
&(0,2,2,1,2),(0,2,1,2,1),(0,0,0,2,1),(0,0,1,2,2),(2,0,1,1,0),(1,2,2,0,0),\\
&(1,0,2,1,2),(1,2,2,0,2),(2,2,2,0,2),(0,0,2,0,1)\},\\
B_7 = \{&(0,2,0,2,2),(2,2,0,2,0),(0,2,0,0,1),(2,2,2,0,1),(1,2,2,2,1),(1,0,0,0,1),\\
&(0,2,0,1,2),(0,1,1,0,2),(1,0,1,1,0),(0,1,2,0,1),(2,1,1,1,1),(0,1,0,1,1),\\
&(1,1,0,1,0),(0,1,0,0,2),(1,1,1,0,2),(2,1,1,1,2),(2,0,0,0,2),(0,1,0,2,1),\\
&(0,2,2,0,1),(2,0,2,2,0),(0,2,1,0,2),(1,2,2,2,2)\},\\
B_8 = \{&(1,0,2,2,1),(1,2,1,1,0),(0,2,1,2,2),(2,1,0,0,1),(1,2,1,2,0),(2,0,0,1,0),\\
&(1,0,2,2,2),(0,0,1,2,0),(1,1,1,2,0),(2,1,2,0,1),(1,0,1,1,2),(2,0,1,1,2),
\end{aligned}$$

$$
\begin{aligned}
&(2,1,2,2,0),(0,1,2,1,1),(1,2,0,0,2),(2,1,2,1,0),(1,0,0,2,0),(2,0,1,1,1),\\
&(0,0,2,1,0),(2,2,2,1,0),(1,2,1,0,2),(2,0,2,2,1)\},\\
B_9 = \{&(2,1,1,0,2),(0,2,1,1,1),(1,2,1,0,0),(1,1,2,0,1),(0,2,1,1,0),(2,1,1,2,0),\\
&(0,2,2,0,0),(0,2,2,1,0),(2,0,1,0,1),(1,2,0,2,2),(0,0,0,2,2),(1,2,2,0,1),\\
&(0,1,2,2,2),(2,1,2,0,0),(2,2,1,0,2),(0,1,2,2,0),(1,2,2,1,0),(0,1,1,0,0),\\
&(0,1,1,2,0),(1,0,2,0,2),(2,1,0,1,1),(0,0,0,1,1)\},\\
B_{10} = \{&(2,1,2,1,1),(0,1,0,0,0),(0,2,1,1,2),(0,0,1,1,0),(2,0,2,0,2),(0,2,0,1,0),\\
&(2,2,1,1,2),(2,2,1,1,1),(0,1,1,0,1),(2,2,1,0,0),(2,2,2,0,0),(1,2,1,2,2),\\
&(0,2,0,0,0),(0,1,2,2,1),(0,0,2,2,0),(1,0,1,0,1),(0,1,0,2,0),(1,1,2,2,1),\\
&(1,1,2,2,2),(0,2,2,0,2),(1,1,2,0,0),(1,1,1,0,0)\},\\
B_{11} = \{&(1,2,0,2,1),(1,2,0,1,2),(1,2,1,0,1),(0,2,2,1,1),(1,1,1,1,1),(2,1,0,2,0),\\
&(0,0,2,2,2),(2,2,1,2,1),(2,2,0,2,2),(1,1,0,2,1),(1,0,2,0,0),(2,1,0,1,2),\\
&(2,1,0,2,1),(2,1,2,0,2),(0,1,1,2,2),(2,2,2,2,2),(1,2,0,1,0),(0,0,1,1,1),\\
&(1,1,2,1,2),(1,1,0,1,1),(2,2,0,1,2),(2,0,1,0,0)\}.
\end{aligned}
$$

4 An exponent bound and its application

In this section, we present an exponent bound on a group G containing a (v,m,k,λ)-SEDF, and use it to prove nonexistence results for the case $m=2$.

Let H be a subgroup of an abelian group G, and let $\rho: G \to H$ be the canonical epimorphism. Each $\widetilde{\chi} \in \widehat{H}$ induces a lifting character $\chi \in \widehat{G}$ satisfying $\chi(g) = \widetilde{\chi}(\rho(g))$ for every $g \in G$. From now on, we shall use G_p to denote the Sylow p-subgroup of the group G, where p is a prime. For a positive integer n, we use ζ_n to denote the primitive n-th complex root of unity $e^{2\pi i/n}$.

We begin with two preparatory lemmas.

Lemma 4.1. *Let p be a prime, let G be an abelian group of order v with $\exp(G_p) = p^e$, and let H be a cyclic p-subgroup of G of order p^e. Let $\rho: G \to H$ be the canonical epimorphism, and let D be a subset of G. Then for each $\widetilde{\chi}$ that is a generator of $\widehat{H}$, we have $\widetilde{\chi}(\rho(D)) = \sum_{i=0}^{p^e-p^{e-1}-1} c_i \zeta_{p^e}^i$, where each c_i is an integer satisfying $-\frac{v}{p^e} \le c_i \le \frac{v}{p^e}$.*

Proof. Since $\widetilde{\chi}$ is a generator of $\widehat{H}$, there is a generator h of H for which $\widetilde{\chi}(h) = \zeta_{p^e}$. Write $\rho(D) = \sum_{i=0}^{p^e-1} d_i h^i \in \mathbb{Z}[H]$, where $0 \le d_i \le \frac{v}{p^e}$ for each i, and then $\widetilde{\chi}(\rho(D)) = \sum_{i=0}^{p^e-1} d_i \zeta_{p^e}^i$. Expanding i as $jp^{e-1}+\ell$, we have

$$
\begin{aligned}
\widetilde{\chi}(\rho(D)) &= \sum_{\ell=0}^{p^{e-1}-1} \sum_{j=0}^{p-1} d_{jp^{e-1}+\ell}\, \zeta_{p^e}^{jp^{e-1}+\ell} \\
&= \sum_{\ell=0}^{p^{e-1}-1} \left(\sum_{j=0}^{p-2} d_{jp^{e-1}+\ell}\, \zeta_{p^e}^{jp^{e-1}+\ell} + d_{(p-1)p^{e-1}+\ell}\, \zeta_{p^e}^{(p-1)p^{e-1}+\ell} \right) \\
&= \sum_{\ell=0}^{p^{e-1}-1} \left(\sum_{j=0}^{p-2} d_{jp^{e-1}+\ell}\, \zeta_{p^e}^{jp^{e-1}+\ell} - d_{(p-1)p^{e-1}+\ell} \sum_{j=0}^{p-2} \zeta_{p^e}^{jp^{e-1}+\ell} \right) \\
&= \sum_{\ell=0}^{p^{e-1}-1} \sum_{j=0}^{p-2} (d_{jp^{e-1}+\ell} - d_{(p-1)p^{e-1}+\ell}) \zeta_{p^e}^{jp^{e-1}+\ell}
\end{aligned}
$$

$$= \sum_{i=0}^{(p-1)p^{e-1}-1} c_i \zeta_{p^e}^i$$

by putting $jp^{e-1} + \ell = i$, and each c_i is an integer satisfying $-\frac{v}{p^e} \le c_i \le \frac{v}{p^e}$. □

A prime p is *self-conjugate* modulo n if there is an integer j for which $p^j \equiv -1 \pmod{n_p}$, where n_p is the largest divisor of n that is not divisible by p. For $X \in \mathbb{Z}[\zeta_n]$, we use (X) to denote the principal idea generated by X in $\mathbb{Z}[\zeta_n]$.

Lemma 4.2. *Let p and q be primes, let q be a primitive root mod p^e, and let $q^f \mid\mid u$ for some positive integer f. Suppose that $X, X' \in \mathbb{Z}[\zeta_{p^e}]$ satisfy $X\overline{X'} = u$. Then either $X \equiv 0 \pmod{q^{\lceil f/2 \rceil}}$ or $X' \equiv 0 \pmod{q^{\lceil f/2 \rceil}}$. Furthermore, if $X = X'$, then f is even.*

Proof. Since $X\overline{X'} = u$ and $q^f \mid\mid u$, we have $X\overline{X'} \equiv 0 \pmod{q^f}$. Now q is a primitive root mod p^e, so (q) is a prime ideal in $\mathbb{Z}[\zeta_{p^e}]$ [16, Chapter 13, Theorem 2], which we denote by $\mathcal{Q}$. Hence

$$X\overline{X'} \equiv 0 \pmod{\mathcal{Q}^f}$$

and so

$$\mathcal{Q}^f \mid (X)(\overline{X'}).$$

Therefore either $\mathcal{Q}^{\lceil f/2 \rceil} \mid (X)$ or $\mathcal{Q}^{\lceil f/2 \rceil} \mid (X')$, and so either $X \equiv 0 \pmod{q^{\lceil f/2 \rceil}}$ or $X' \equiv 0 \pmod{q^{\lceil f/2 \rceil}}$.

Now suppose $X = X'$, so that $\mathcal{Q}^{\lceil f/2 \rceil} \mid (X)$. Since q is a primitive root mod p^e, we have that q is self-conjugate modulo p^e. This implies $\mathcal{Q}$ is invariant under complex conjugation [5, Chapter VI, Corollary 15.5], so that $\mathcal{Q}^{\lceil f/2 \rceil} \mid (\overline{X})$ and therefore $\mathcal{Q}^{2\lceil f/2 \rceil} \mid (X)(\overline{X})$. But $q^f \mid\mid u$, so $\mathcal{Q}^f \mid\mid (X)(\overline{X})$. Therefore f is even. □

We now prove the following exponent bound.

Theorem 4.3. *Suppose there exists a (v, m, k, λ)-SEDF in a group G. Let p and q be primes such that $p^d \mid\mid v$ and $q^f \mid\mid \lambda$ for some positive integers d and f, and suppose that q is a primitive root mod p^d. Let G_p be the Sylow p-subgroup of G. Then*

$$\exp(G_p) \le v/q^{\lceil f/2 \rceil}.$$

Proof. Let $\{D_1, D_2, \ldots, D_m\}$ be the SEDF, and let $D = \bigcup_{i=1}^m D_i$. Let $\exp(G_p) = p^e$, let H be a cyclic p-subgroup of G of order p^e, and let $\rho : G \to H$ be the canonical epimorphism. Let $\widetilde{\chi}$ be a generator of $\widehat{H}$, and let χ be the associated lifting character on G. Then

$$\widetilde{\chi}(\rho(D_1))\overline{\widetilde{\chi}(\rho(D - D_1))} = \chi(D_1)\overline{\chi(D - D_1)} = -\lambda \tag{4.1}$$

by (2.1). Now q is a primitive root mod p^d, so q is also a primitive root mod p^e [16, Chapter 4, Lemma 3]. Apply Lemma 4.2 with $X = \widetilde{\chi}(\rho(D_1))$ and $X' = \widetilde{\chi}(\rho(D - D_1))$ and $u = -\lambda$ to show that there is a subset D' of G (either D_1 or $D \setminus D_1$) for which

$$\widetilde{\chi}(\rho(D')) \equiv 0 \pmod{q^{\lceil f/2 \rceil}}.$$

Since $\widetilde{\chi}(\rho(D')) \in \mathbb{Z}[\zeta_{p^e}]$, we may write $\widetilde{\chi}(\rho(D')) = q^{\lceil f/2 \rceil} \sum_{i=0}^{p^e - p^{e-1}-1} d_i \zeta_{p^e}^i$, where each d_i is an integer (because $\{1, \zeta_{p^e}, \zeta_{p^e}^2, \ldots, \zeta_{p^e}^{p^e - p^{e-1}-1}\}$ is an integral basis of $\mathbb{Z}[\zeta_{p^e}]$). But by Lemma 4.1 we also have $\widetilde{\chi}(\rho(D')) = \sum_{i=0}^{p^e - p^{e-1}-1} c_i \zeta_{p^e}^i$, where each c_i is an integer satisfying $-\frac{v}{p^e} \le c_i \le \frac{v}{p^e}$, and so $q^{\lceil f/2 \rceil} d_i = c_i$ for each i. By (4.1) not every d_i is 0, and therefore $q^{\lceil f/2 \rceil} \le v/p^e$, or equivalently $p^e \le v/q^{\lceil f/2 \rceil}$. □

Very few nonexistence results for a nontrivial (v, m, k, λ)-SEDF with $m = 2$ are known. We now illustrate the use of Theorem 4.3 by ruling out several families of such parameter sets. When $m = 2$ and k is prime, the existence question is already answered: we must have $\lambda = 1$ [15, Lemma 3.4], and then by Proposition 1.2 the parameters have the form $(k^2 + 1, 2, k, 1)$. We therefore consider $m = 2$ and $k = p_1p_2$ in Theorem 4.4 below, where p_1, p_2 are distinct primes and $p_1 < p_2$. The case $\lambda = 1$ is dealt with in Proposition 1.2, and the cases $\lambda \geq p_1p_2$ are ruled out by Proposition 1.3 (6). In view of the counting relation $p_1^2p_2^2 = \lambda(v - 1)$ given by (1.3), the remaining cases are $\lambda \in \{p_1, p_2, p_1^2\}$.

Theorem 4.4. *Let p_1 and p_2 be distinct primes with $p_1 < p_2$.*

(1) Let p be a prime such that $p^d \mid\mid p_1p_2^2 + 1$ for some positive integer d. If p_1 is a primitive root mod p^d and $p_2^2 + 1 \leq p$, then a $(p_1p_2^2 + 1, 2, p_1p_2, p_1)$-SEDF does not exist.

(2) Let p be a prime such that $p^d \mid\mid p_1^2p_2 + 1$ for some positive integer d. If p_2 is a primitive root mod p^d and $p_1^2 + 1 \leq p$, then a $(p_1^2p_2 + 1, 2, p_1p_2, p_2)$-SEDF does not exist.

(3) Let p be a prime such that $p^d \mid\mid p_2^2 + 1$ for some positive integer d. If p_1 is a primitive root mod p^d and $p_2^2 + 1 < p_1p$, or if p_1 is a primitive root mod p and $p_2^2 + 1 = p_1p$, then a $(p_2^2 + 1, 2, p_1p_2, p_1^2)$-SEDF does not exist.

Proof. Parts (1), (2) and the first part of (3) are each direct applications of Theorem 4.3, whereas the second part of (3) requires additional arguments; we give the proof for both parts of (3). Suppose, for a contradiction, that $\{D_1, D_2\}$ is a $(p_2^2 + 1, 2, p_1p_2, p_1^2)$-SEDF in a group G of order $p_2^2 + 1$.

If p_1 is a primitive root mod p^d and $p_2^2 + 1 < p_1p$, then by Theorem 4.3 we have $\exp(G_p) \leq \frac{p_2^2+1}{p_1} < p$. This contradicts that p is a prime divisor of $|G| = p_2^2 + 1$.

If p_1 is a primitive root mod p and $p_2^2 + 1 = p_1p$, then $G = \mathbb{Z}_{p_1} \times \mathbb{Z}_p$. Let $\rho : G \to \mathbb{Z}_p$ be the canonical epimorphism. Let $\widetilde{\chi}$ be a generator of $\widehat{\mathbb{Z}_p}$. Then

$$\widetilde{\chi}(\rho(D_1))\overline{\widetilde{\chi}(\rho(D_2))} = -p_1^2$$

by (2.1), so by Lemma 4.2 we may choose D' to be one of D_1 and D_2 so that $\widetilde{\chi}(\rho(D')) \equiv 0 \pmod{p_1}$. Since $\widetilde{\chi}$ is a generator of $\widehat{\mathbb{Z}_p}$, there is a generator h of $\mathbb{Z}_p$ for which $\widetilde{\chi}(h) = \zeta_p$. Write $\rho(D') = \sum_{i=0}^{p-1} d_ih^i$, where $0 \leq d_i \leq p_1$ for each i, and then

$$\begin{aligned}\widetilde{\chi}(\rho(D')) &= \sum_{i=0}^{p-1} d_i\zeta_p^i \\ &= \sum_{i=0}^{p-2}(d_i - d_{p-1})\zeta_p^i.\end{aligned}$$

Since $\widetilde{\chi}(\rho(D')) \equiv 0 \pmod{p_1}$, we have $p_1 \mid d_i - d_{p-1}$ for each i. Using $0 \leq d_i \leq p_1$ for each i, we distinguish two cases:

Case 1: $d_i \in \{0, p_1\}$ for each i satisfying $0 \leq i \leq p - 1$. This gives $\rho(D') = p_1 \sum_{i \in I} h^i$ for some subset I of $\{0, 1, \ldots, p - 1\}$, which implies that D' is a union of cosets of $\mathbb{Z}_{p_1}$. But then for a character $\chi \in \widehat{G}$ which is nonprincipal on $\mathbb{Z}_{p_1}$ we have $\chi(D') = 0$, contradicting (2.1) because $D' = D_1$ or D_2.

Case 2: $d_0 = d_1 = \cdots = d_{p-1}$. Then p divides $|D'| = p_1p_2$, so either $p = p_1$ or $p = p_2$. Both of these contradict the given conditions on p, p_1, p_2.

□

Remark 4.5. *For example, Theorem* 4.4 *rules out the existence of a* (v,m,k,λ)*-SEDF for*

$$(v,m,k,\lambda) \in \{(19,2,6,2),(26,2,10,4),(46,2,15,5),(118,2,39,13),(122,2,22,4),\\ (154,2,51,17),(172,2,57,19)\}.$$

Theorem 4.3 *rules out further parameter sets not excluded by Theorem* 4.4 *(for which* k *is not the product of two distinct primes), including*

$$(v,m,k,\lambda) \in \{(37,2,12,4),(101,2,20,4),(101,2,30,9),(101,2,40,16),(122,2,44,16),\\ (127,2,42,14),(129,2,48,18),(163,2,18,2),(163,2,36,8),(163,2,54,18),\\ (163,2,72,32),(177,2,44,11),(181,2,60,20),(197,2,28,4),(197,2,42,9),\\ (197,2,56,16),(197,2,70,25),(197,2,84,36)\}.$$

All known examples of a nontrivial $(v,2,k,\lambda)$*-SEDF have* v *a prime power, except for those specified in Proposition* 1.1 (1) *and* (2)*. The only cases for a* $(v,2,k,\lambda)$*-SEDF with* $v \le 50$ *that remain open are*

$$(v,m,k,\lambda) \in \{(28,2,9,3),(33,2,8,2),(49,2,12,3),(50,2,14,4),(50,2,21,9)\}.$$

5 SEDFs with $m > 2$

Throughout this section, we suppose that $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v,m,k,λ)-SEDF in a group G with $m > 2$, and write $D = \bigcup_{i=1}^m D_i$. From (2.7), in order for ℓ_χ^+ and ℓ_χ^- to be integers we must have

$$\sqrt{1 + \tfrac{4\lambda}{|\chi(D)|^2}} = \frac{b_\chi}{a_\chi} \quad \text{for each } \chi \in \widehat{G}^N, \quad \text{where } a_\chi, b_\chi \in \mathbb{Z} \text{ and } b_\chi > a_\chi > 0 \text{ and } \gcd(a_\chi, b_\chi) = 1. \tag{5.1}$$

Then (2.7) becomes

$$\ell_\chi^+ = \frac{m}{2} - \frac{a_\chi(m-2)}{2b_\chi}, \quad \ell_\chi^- = \frac{m}{2} + \frac{a_\chi(m-2)}{2b_\chi} \quad \text{for each } \chi \in \widehat{G}^N. \tag{5.2}$$

It is shown in [3, Lemma 3.3] and [21, Lemma 3.5] that $(\ell_\chi^+, \ell_\chi^-) \notin \{(0,m),(1,m-1),(\frac{m}{2},\frac{m}{2})\}$ for $m > 2$; this is an immediate consequence of (5.2). Rewrite the expressions (2.6) for α_χ^+ and α_χ^- using (5.1), and then substitute into (2.8) to obtain

$$\{\chi(D_j) \mid 1 \le j \le m\} = \Big\{ \frac{a_\chi + b_\chi}{2a_\chi}\chi(D), \frac{a_\chi - b_\chi}{2a_\chi}\chi(D) \Big\} \quad \text{for each } \chi \in \widehat{G}^N. \tag{5.3}$$

Rearrange (5.1) as

$$|\chi(D)|^2 = \frac{4a_\chi^2\lambda}{b_\chi^2 - a_\chi^2} \quad \text{for each } \chi \in \widehat{G}^N,$$

and then combine with (5.3) to give

$$\Big\{ \big(|\chi(D)|^2, |\chi(D_j)|^2\big) \mid 1 \le j \le m \Big\} \\ = \left\{ \left(\frac{4a_\chi^2\lambda}{b_\chi^2 - a_\chi^2}, \frac{(b_\chi + a_\chi)\lambda}{b_\chi - a_\chi} \right), \left(\frac{4a_\chi^2\lambda}{b_\chi^2 - a_\chi^2}, \frac{(b_\chi - a_\chi)\lambda}{b_\chi + a_\chi} \right) \right\} \quad \text{for each } \chi \in \widehat{G}^N. \tag{5.4}$$

We now derive some divisibility conditions on the values of a_χ and b_χ, which restrict the possible values of $|\chi(D)|^2$ and $|\chi(D_j)|^2$ via (5.4).

Lemma 5.1. *Let a_χ, b_χ be defined as in* (5.1) *(with reference to the set $D = \bigcup_{i=1}^m D_i$ associated with a nontrivial (v, m, k, λ)-SEDF $\{D_1, D_2, \ldots, D_m\}$ in a group G with $m > 2$). Then*

(1) $2b_\chi \mid b_\chi m - a_\chi(m-2)$, *and* $b_\chi \mid m-2$

(2) $(b_\chi - a_\chi) \mid (b_\chi + a_\chi)\lambda$, *and* $(b_\chi + a_\chi) \mid (b_\chi - a_\chi)\lambda$

(3) $(b_\chi^2 - a_\chi^2) \mid 4\lambda$, *and if $b_\chi + a_\chi$ is odd then* $(b_\chi^2 - a_\chi^2) \mid \lambda$.

Proof.

(1) Since ℓ_χ^+ is an integer, by (5.2) we have $2b_\chi \mid b_\chi m - a_\chi(m-2)$. Therefore $b_\chi \mid a_\chi(m-2)$, and since $\gcd(a_\chi, b_\chi) = 1$ we have $b_\chi \mid m-2$.

(2) $|\chi(D_j)|^2$ is an algebraic integer, and by (5.4) also takes both the rational values $\frac{(b_\chi+a_\chi)\lambda}{b_\chi-a_\chi}$ and $\frac{(b_\chi-a_\chi)\lambda}{b_\chi+a_\chi}$ as j ranges over $\{1, 2, \ldots, m\}$. Therefore $\frac{(b_\chi+a_\chi)\lambda}{b_\chi-a_\chi}$ and $\frac{(b_\chi-a_\chi)\lambda}{b_\chi+a_\chi}$ are both integers.

(3) $|\chi(D)|^2$ is an algebraic integer, and by (5.4) is also the rational number $\frac{4a_\chi^2\lambda}{b_\chi^2-a_\chi^2}$. Therefore $\frac{4a_\chi^2\lambda}{b_\chi^2-a_\chi^2}$ is an integer, which implies $(b_\chi^2-a_\chi^2) \mid 4\lambda$. If $b_\chi+a_\chi$ is odd, then $\gcd(b_\chi-a_\chi, b_\chi+a_\chi) = \gcd(2b_\chi, b_\chi+a_\chi) = \gcd(b_\chi, b_\chi+a_\chi) = 1$ so that from part (2) we obtain $(b_\chi - a_\chi) \mid \lambda$ and $(b_\chi + a_\chi) \mid \lambda$ and therefore $(b_\chi^2 - a_\chi^2) \mid \lambda$.

□

Using Lemma 5.1, we recover the result of Proposition 1.3 (1) as Corollary 5.2, and obtain new restrictions for $m \in \{5, 6\}$ as Corollary 5.3.

Corollary 5.2. *A nontrivial (v, m, k, λ)-SEDF does not exist for $m \in \{3, 4\}$.*

Corollary 5.3. *Let a_χ, b_χ be defined as in* (5.1).

(1) If there exists a nontrivial $(v, 5, k, \lambda)$-SEDF in a group G, then $(a_\chi, b_\chi) = (1, 3)$ and $2 \mid \lambda$ for each $\chi \in \widehat{G}^N$.

(2) If there exists a nontrivial $(v, 6, k, \lambda)$-SEDF in a group G, then $(a_\chi, b_\chi) = (1, 2)$ and $3 \mid \lambda$ for each $\chi \in \widehat{G}^N$.

Motivated by Corollary 5.3, we say that a nontrivial (v, m, k, λ)-SEDF with $m > 2$ for which (a_χ, b_χ) takes a constant value (a, b) for all $\chi \in \widehat{G}^N$ has the *simple character value property* with respect to (a, b). In the following subsection we obtain restrictions on SEDFs having this property. In particular, for $m = 5$ and for $m = 6$ we obtain asymptotic nonexistence results for a family of SEDFs, each of which must have this property with respect to a fixed (a, b) by Corollary 5.3.

5.1 The simple character value property

As above, suppose that $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, m, k, λ)-SEDF in a group G with $m > 2$, and write $D = \bigcup_{i=1}^m D_i$. Suppose further that $\{D_1, D_2, \ldots, D_m\}$ has the simple character value property with respect to (a, b). Then by (5.3), we may partition $\widehat{G}^N$ (with respect to D_1) into the disjoint union of the sets

$$\widehat{G}^+ = \{\chi \in \widehat{G}^N \mid \chi(D_1) = \frac{a+b}{2a}\chi(D)\}, \tag{5.5}$$

$$\widehat{G}^- = \{\chi \in \widehat{G}^N \mid \chi(D_1) = \frac{a-b}{2a}\chi(D)\}, \tag{5.6}$$

and from the definition (2.3), $\widehat{G}$ is the disjoint union $\{\chi_0\} \cup \widehat{G}^0 \cup \widehat{G}^+ \cup \widehat{G}^-$. By (2.2) and (5.4), we then obtain the character values in Table 5.1.

$\chi \in \widehat{G}$	$\lvert\chi(D)\rvert^2$	$\chi(D_1)$	$\lvert\chi(D_1)\rvert^2$
$\chi = \chi_0$	k^2m^2	k	k^2
$\chi \in \widehat{G}^0$	0		λ
$\chi \in \widehat{G}^+$	$\dfrac{4a^2\lambda}{b^2-a^2}$	$\dfrac{a+b}{2a}\chi(D)$	$\dfrac{(b+a)\lambda}{b-a}$
$\chi \in \widehat{G}^-$	$\dfrac{4a^2\lambda}{b^2-a^2}$	$\dfrac{a-b}{2a}\chi(D)$	$\dfrac{(b-a)\lambda}{b+a}$

Table 5.1: Character sums for an SEDF with $m > 2$, having the simple character value property with respect to (a, b)

We now determine the size of the sets $\widehat{G}^0$, $\widehat{G}^+$, $\widehat{G}^-$.

Theorem 5.4. *Suppose $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, m, k, λ)-SEDF in a group G with $m > 2$, having the simple character value property with respect to (a, b). Then the sizes of the sets $\widehat{G}^0$, $\widehat{G}^+$, $\widehat{G}^-$ (defined as in (2.3), (5.5), (5.6) with reference to the sets D_1 and $D = \bigcup_{i=1}^m D_i$) are*

$$\begin{aligned} |\widehat{G}^0| &= (v-1)\left(1 - \frac{(b^2-a^2)(v-km)m}{4a^2k(m-1)}\right), \\ |\widehat{G}^+| &= \frac{(v-1)(v-km)(b^2-a^2)((b-a)m+2a)}{8a^2bk(m-1)}, \\ |\widehat{G}^-| &= \frac{(v-1)(v-km)(b^2-a^2)((b+a)m-2a)}{8a^2bk(m-1)}, \end{aligned} \tag{5.7}$$

and each of $|\widehat{G}^0|, |\widehat{G}^+|, |\widehat{G}^-|$ is a non-negative integer and $|\widehat{G}^+| + |\widehat{G}^-| > 0$.

Proof. Each of $|\widehat{G}^0|, |\widehat{G}^+|, |\widehat{G}^-|$ is a non-negative integer by definition, and $|\widehat{G}^+| + |\widehat{G}^-| = |\widehat{G}^N| > 0$ by Lemma 2.2. Write $DD^{(-1)} = \sum_{g \in G} c_g g \in \mathbb{Z}[G]$. From Proposition 2.1,

$$c_1 = \frac{1}{v} \sum_{\chi \in \widehat{G}} |\chi(D)|^2.$$

The left side $c_1 = |D| = km$ is the coefficient of the identity in the expression $DD^{(-1)}$, and the right side can be evaluated using Table 5.1 to give

$$km = \frac{1}{v}\left(k^2m^2 + \big(v - 1 - |\widehat{G}^0|\big)\frac{4a^2\lambda}{b^2-a^2}\right).$$

Substitute for λ from the counting relation (1.3) to obtain the required expression for $|\widehat{G}^0|$.

Similarly, write $D_1 D_1^{(-1)} = \sum_{g \in G} d_g g \in \mathbb{Z}[G]$ and use Proposition 2.1 and Table 5.1 to give

$$k = \frac{1}{v}\Big(k^2 + |\widehat{G}^0|\,\lambda + |\widehat{G}^+|\,\frac{(b+a)\lambda}{b-a} + |\widehat{G}^-|\,\frac{(b-a)\lambda}{b+a}\Big).$$

We now obtain the required expressions for $|\widehat{G}^+|$ and $|\widehat{G}^-|$ using (5.7) and the counting condition $|\widehat{G}^0| + |\widehat{G}^+| + |\widehat{G}^-| = v - 1$. □

We obtain the following asymptotic nonexistence result from Theorem 5.4.

Theorem 5.5. *Let m, λ, a, b be fixed positive integers, where $m > 2$ and $b > a$ and $\gcd(a, b) = 1$. Then for all sufficiently large k, there does not exist a nontrivial (v, m, k, λ)-SEDF having the simple character value property with respect to (a, b).*

Proof. Apply the condition $|\widehat{G}^0| \geq 0$ to (5.7), and rearrange to give the inequality

$$\frac{v}{k} \leq m + \frac{4a^2(m-1)}{m(b^2 - a^2)}.$$

Since m and λ are fixed, the counting relation (1.3) shows that v grows like k^2 as k increases. Therefore for all sufficiently large k, the inequality in v/k does not hold. □

As a consequence of Corollary 5.3 and Theorem 5.5, we obtain the following asymptotic nonexistence result for $m \in \{5, 6\}$.

Corollary 5.6. *Let λ be a fixed positive integer. Then for all sufficiently large k, there does not exist a nontrivial $(v, 5, k, \lambda)$-SEDF and there does not exist a nontrivial $(v, 6, k, \lambda)$-SEDF.*

We can obtain results similar to Corollary 5.6 for values of m greater than 6. For example, suppose there exists a nontrivial $(v, 7, k, \lambda)$-SEDF. From Lemma 5.1 we find that $\lambda \bmod 12 \in \{0, 4, 6, 8\}$, and that the SEDF has the simple character value property with respect to $(1, 5)$ if $\lambda \bmod 12 = 6$ and with respect to $(3, 5)$ if $\lambda \bmod 12 \in \{4, 8\}$. Therefore for fixed λ for which $\lambda \bmod 12 \neq 0$, for all sufficiently large k there does not exist a nontrivial $(v, 7, k, \lambda)$-SEDF. Likewise, for fixed λ for which $\lambda \bmod 10 \neq 0$, for all sufficiently large k there does not exist a nontrivial $(v, 8, k, \lambda)$-SEDF.

We derive further divisibility conditions on the SEDF parameters in Theorem 5.9 below. We first require two number-theoretic lemmas.

Lemma 5.7. [14, Lemma 2.3] *Let p be a prime and let e be a positive integer. Let $\sigma = \sum_{i=0}^{p^e-1} c_i \zeta_{p^e}^i$, where each $c_i \in \mathbb{Z}$. Then $\sigma = 0$ if and only if $c_i = c_j$ for all i and j satisfying $i \equiv j \pmod{p^{e-1}}$.*

Lemma 5.8. *Let p be a prime and H be a p-group. Let $E = \sum_{h \in H} c_h h \in \mathbb{Z}[H]$, where each $c_h \geq 0$ and $\sum_{h \in H} c_h = u$. Suppose there is an integer ℓ and a character $\chi \in \widehat{H}$ for which $|\chi(E)|^2 = \ell$. Then $u^2 + (p-1)\ell = pr$ for some integer $r \geq u$.*

Proof. Let $p^e = \exp(H)$. Then

$$\begin{aligned} \ell &= |\chi(E)|^2 \\ &= \sum_{h,j \in H} c_h c_j \chi(h)\overline{\chi(j)} \\ &= \sum_{i=0}^{p^e-1} d_i \zeta_{p^e}^i, \end{aligned} \tag{5.8}$$

where $d_i = \sum_{h,j\in H:\chi(h)\overline{\chi(j)}=\zeta_{p^e}^i} c_h c_j$. Each d_i is a non-negative integer, and

$$\begin{aligned}\sum_{i=0}^{p^e-1} d_i &= \sum_{h,j\in H} c_h c_j \\ &= u^2. \end{aligned} \tag{5.9}$$

Subtract ℓ from both sides of (5.8), and deduce from Lemma 5.7 that

$$d_0 - \ell = d_{p^{e-1}} = d_{2p^{e-1}} = \cdots = d_{(p-1)p^{e-1}}$$

and

$$d_j = d_{p^{e-1}+j} = d_{2p^{e-1}+j} = \cdots = d_{(p-1)p^{e-1}+j} \quad \text{for each } j \text{ satisfying } 1 \le j \le p^{e-1}-1.$$

Substitute into (5.9) to obtain

$$p\sum_{i=0}^{p^{e-1}-1} d_i - (p-1)\ell = u^2,$$

so that $u^2 + (p-1)\ell = pr$ where r is an integer satisfying

$$r = \sum_{i=0}^{p^{e-1}-1} d_i \ge d_0 = \sum_{h,j\in H:\chi(h)=\chi(j)} c_h c_j \ge \sum_{h\in H} c_h^2 \ge \sum_{h\in H} c_h = u.$$

□

Theorem 5.9. *Suppose $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, m, k, λ)-SEDF in a group G with $m > 2$, having the simple character value property with respect to (a, b), and let p be a prime divisor of v. Then either the following both hold:*

(1a) $|G_p|$ divides km,

(1b) $k^2 + (|G_p| - 1)\lambda = |G_p| r_1$ for some integer $r_1 \ge k$.

or the following all hold:

(2a) $k^2m^2 + (p-1)\frac{4a^2\lambda}{b^2-a^2} = pr_2$ for some integer $r_2 \ge km$,

(2b) $k^2 + (p-1)\frac{(b-a)\lambda}{b+a} = pr_3$ for some integer $r_3 \ge k$,

(2c) $k^2 + (p-1)\frac{(b+a)\lambda}{b-a} = pr_4$ for some integer r_4.

Proof. Let $\rho : G \to G_p$ be the canonical epimorphism, and let $D = \bigcup_{i=1}^m D_i$. For each nonprincipal character $\widetilde{\chi} \in \widehat{G_p}$ and its associated lifting character $\chi \in \widehat{G}$, Table 5.1 gives

$$\begin{aligned}&\big(|\widetilde{\chi}(\rho(D))|^2, |\widetilde{\chi}(\rho(D_1))|^2\big) \\ &= \big(|\chi(D)|^2, |\chi(D_1)|^2\big) = \begin{cases} \left(\dfrac{4a^2\lambda}{b^2-a^2}, \dfrac{(b+a)\lambda}{b-a}\right) \text{ or } \left(\dfrac{4a^2\lambda}{b^2-a^2}, \dfrac{(b-a)\lambda}{b+a}\right) & \text{for } \chi \in \widehat{G}^N, \\ (0, \lambda) & \text{for } \chi \in \widehat{G}^0. \end{cases}\end{aligned} \tag{5.10}$$

Case 1: $\widetilde{\chi}(\rho(D)) = 0$ for every nonprincipal character $\widetilde{\chi} \in \widehat{G_p}$. Apply Proposition 2.1 with $A = \rho(D)$ to obtain $\rho(D) = \frac{km}{|G_p|} G_p$, giving (1a). By (5.10), we have $|\widetilde{\chi}(\rho(D_1))|^2 = \lambda$ for every nonprincipal character $\widetilde{\chi} \in \widehat{G_p}$. Apply Proposition 2.1 with $A = \rho(D_1)\rho(D_1)^{(-1)} = \sum_{g \in \widehat{G_p}} c_g g$ to obtain $c_1 = \frac{1}{|G_p|}\big(k^2 + (|G_p| - 1)\lambda\big)$ and note that $c_1 \geq |D_1| = k$, giving (1b).

Case 2: $\widetilde{\chi}(\rho(D)) \neq 0$ for some nonprincipal character $\widetilde{\chi} \in \widehat{G_p}$. By (5.10), this $\widetilde{\chi}$ satisfies $|\widetilde{\chi}(\rho(D))|^2 = \frac{4a^2\lambda}{b^2-a^2}$, which is an integer by Lemma 5.1 (3). Apply Lemma 5.8 with $(H, E) = (G_p, \rho(D))$ and $u = |D| = km$ to give (2a). By (5.10), this $\widetilde{\chi}$ also satisfies $|\widetilde{\chi}(\rho(D_1))|^2 = \frac{(b+a)\lambda}{b-a}$ or $\frac{(b-a)\lambda}{b+a}$. Then by (5.4) there is some $j \neq 1$ for which

$$\big\{|\widetilde{\chi}(\rho(D_1))|^2, |\widetilde{\chi}(\rho(D_j))|^2\big\} = \left\{\frac{(b+a)\lambda}{b-a}, \frac{(b-a)\lambda}{b+a}\right\},$$

and both values are integers by Lemma 5.1 (2). Apply Lemma 5.8 with $(H, E) = (G_p, \rho(D_1))$ and with $(H, E) = (G_p, \rho(D_j))$ to give (2b) and (2c).

□

By Proposition 1.3 (2), we know that a nontrivial (v, m, k, λ)-SEDF does not exist when v is prime and $m > 2$. We now prove a nonexistence result when v is a prime power and $m > 2$.

Theorem 5.10. *Let $v = p^s$ for an odd prime p, and suppose 2 is a primitive root mod p^s. Suppose $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, m, k, λ)-SEDF in a group G with $m > 2$, having the simple character value property with respect to (a, b). Then a and b are both odd.*

Proof. Suppose, for a contradiction, that a and b are not both odd. Since $\gcd(a, b) = 1$, we therefore have $b + a$ odd and so $(b^2 - a^2) \mid \lambda$ by Lemma 5.1 (3). We shall show that this implies km and $v - km$ are both even, contradicting that $v = p^s$ for odd p.

Write $D = \bigcup_{i=1}^{m} D_i$ and use Table 5.1 to give

$$|\chi(D)|^2 = \begin{cases} \frac{4a^2\lambda}{b^2-a^2} & \text{if } \chi \in \widehat{G}^N, \\ 0 & \text{if } \chi \in \widehat{G}^0. \end{cases} \tag{5.11}$$

Since $(b^2 - a^2) \mid \lambda$, this gives

$$\chi(D)\overline{\chi(D)} \equiv 0 \pmod 2 \quad \text{for all nonprincipal } \chi \in \widehat{G}.$$

Now 2 is a primitive root mod p^s, so 2 is also a primitive root mod p^e. Therefore by Lemma 4.2 we have

$$\chi(D) \equiv 0 \pmod 2 \quad \text{for all nonprincipal } \chi \in \widehat{G}. \tag{5.12}$$

By taking a translate of D if necessary, we may assume that $1 \notin D$. Write $D = \sum_{g \in G} d_g g \in \mathbb{Z}[G]$. From Proposition 2.1,

$$0 = vd_1 = \sum_{\chi \in \widehat{G}} \chi(D) = km + \sum_{\chi \in \widehat{G}^N} \chi(D). \tag{5.13}$$

Now $v = p^s$, so $\exp(G) = p^e$ for some integer $e \leq s$. Then using (5.12), for each $\chi \in \widehat{G}^N$ we may write $\chi(D) = 2\sum_{i=0}^{p^e-1} c_{i,\chi}\zeta_{p^e}^i$, where each $c_{i,\chi} \in \mathbb{Z}$. Substitute in (5.13) to give

$$km + 2\sum_{i=0}^{p^e-1} \sum_{\chi \in \widehat{G}^N} c_{i,\chi}\zeta_{p^e}^i = 0.$$

The coefficients of 1 and $\zeta_{p^e}^{p^{e-1}}$ are equal by Lemma 5.7, so

$$km + 2\sum_{\chi\in\widehat{G}^N} c_{0,\chi} = 2\sum_{\chi\in\widehat{G}^N} c_{p^{e-1},\chi}\zeta_{p^e}^{p^{e-1}}.$$

Reduce modulo 2 to show that km is even, as required.

To show that $v - km$ is even, repeat the above analysis with D replaced by $G \setminus D$, noting that $|\chi(G-D)|^2 = |\chi(D)|^2$ for each nonprincipal $\chi \in \widehat{G}$. □

We now illustrate the use of Theorem 5.10 to rule out the existence of an $(81, 6, 12, 9)$-SEDF and a $(6561, 6, 984, 738)$-SEDF.

Example 5.11. *Suppose, for a contradiction, that there exists an* $(81, 6, 12, 9)$*-SEDF or there exists a* $(6561, 6, 984, 738)$*-SEDF. By Corollary 5.3 (2), these SEDFs have the simple character value property with respect to* $(1, 2)$*. Since* 2 *is a primitive root mod* 81 *and mod* 6561*, Theorem 5.10 then gives the contradiction that* 2 *is odd.*

5.2 Further nonexistence results

In this subsection, we extend the analysis of Section 5.1 to the case of an SEDF for which the simple character value property does not necessarily hold. Suppose that $\{D_1, D_2, \ldots, D_m\}$ is a nontrivial (v, m, k, λ)-SEDF in a group G with $m > 2$, and let $D = \bigcup_{i=1}^m D_i$. Suppose that (a_χ, b_χ) as defined in (5.1) takes exactly $t \geq 1$ distinct values as χ ranges over $\widehat{G}^N$, so that

$$\Big\{\sqrt{1 + \tfrac{4\lambda}{|\chi(D)|^2}} \mid \chi \in \widehat{G}^N\Big\} = \Big\{\frac{b_i}{a_i} \mid 1 \leq i \leq t\Big\},$$

where $a_i, b_i \in \mathbb{Z}$ and $b_i > a_i > 0$ and $\gcd(a_i, b_i) = 1$. Define

$$S^+ = \left\{\Big(\frac{4a_i^2\lambda}{b_i^2 - a_i^2}, \frac{(b_i + a_i)\lambda}{b_i - a_i}\Big) \mid 1 \leq i \leq t\right\}, \qquad S^- = \left\{\Big(\frac{4a_i^2\lambda}{b_i^2 - a_i^2}, \frac{(b_i - a_i)\lambda}{b_i + a_i}\Big) \mid 1 \leq i \leq t\right\}.$$

Then from (2.2), (2.3) and (5.4) we have

$$\begin{cases} \big(|\chi(D)|^2, |\chi(D_1)|^2\big) \in S^+ \cup S^- & \text{for } \chi \in \widehat{G}^N, \\ \big(|\chi(D)|^2, |\chi(D_1)|^2\big) = (0, \lambda) & \text{for } \chi \in \widehat{G}^0. \end{cases} \tag{5.14}$$

The following result generalizes Theorem 5.9. The proof, which is omitted, uses (5.14) in a similar manner to the use of (5.10) in the proof of Theorem 5.9.

Theorem 5.12. *Suppose* $\{D_1, D_2, \ldots, D_m\}$ *is a nontrivial* (v, m, k, λ)*-SEDF in a group* G *with* $m > 2$*, let* p *be a prime divisor of* v*. Suppose that* (a_χ, b_χ) *as defined in* (5.1) *takes values in the set* $\{(a_i, b_i) \mid 1 \leq i \leq t\}$ *as* χ *ranges over* $\widehat{G}^N$ *(defined with reference to the set* $D = \bigcup_{i=1}^m D_i$*). Then either the following both hold:*

(1a) $|G_p|$ *divides* km*,*

(1b) $k^2 + (|G_p| - 1)\lambda = |G_p| r_1$ *for some integer* $r_1 \geq k$*.*

or, for some i *satisfying* $1 \leq i \leq t$*, the following all hold:*

(2a) $k^2m^2 + (p-1)\frac{4a_i^2\lambda}{b_i^2 - a_i^2} = pr_{2,i}$ *for some integer* $r_{2,i} \geq km$*,*

(2b) $k^2 + (p-1)\frac{(b_i - a_i)\lambda}{b_i + a_i} = pr_{3,i}$ *for some integer* $r_{3,i} \geq k$,

(2c) $k^2 + (p-1)\frac{(b_i + a_i)\lambda}{b_i - a_i} = pr_{4,i}$ *for some integer* $r_{4,i}$.

We now illustrate the use of Theorem 5.12 to rule out the existence of a (676,26,18,12)-SEDF and a (2401,37,60,54)-SEDF.

Example 5.13. *Suppose, for a contradiction, that there exists a* $(676, 26, 18, 12)$*-SEDF. By Lemma 5.1,*

$$(a_\chi, b_\chi) \in \{(1,2),(1,3)\} \quad \text{for all } \chi \in \widehat{G}^N.$$

For each of these possible values of (a_χ, b_χ)*, both (1a) and (2a) of Theorem 5.12 fail with* $p = 13$*, giving the required contradiction.*

Example 5.14. *Suppose, for a contradiction, that there exists a* $(2401, 37, 60, 54)$*-SEDF. By Lemma 5.1,*

$$(a_\chi, b_\chi) \in \{(1,5),(5,7)\} \quad \text{for all } \chi \in \widehat{G}^N.$$

We cannot have $(a_\chi, b_\chi) = (1,5)$ *for* $\chi \in \widehat{G}^N$*, otherwise both (1a) and (2a) of Theorem 5.12 fail with* $p = 7$*. Therefore* $(a_\chi, b_\chi) = (5,7)$ *for all* $\chi \in \widehat{G}^N$*, and so the SEDF satisfies the simple character value property with respect to* $(5,7)$*. Theorem 5.4 then gives* $|\widehat{G}^0| = \frac{9212}{15}$*, which contradicts that* $|\widehat{G}^0|$ *is an integer.*

We now extend the nonexistence result of Theorem 5.10 for v a prime power and $m > 2$.

Theorem 5.15. *Let* $v = p^s$ *for a prime* p*, and let* a_χ, b_χ *be defined as in* (5.1) *with reference to the set* $D = \bigcup_{i=1}^m D_i$ *associated with a nontrivial* (v, m, k, λ)*-SEDF* $\{D_1, D_2, \ldots, D_m\}$ *in a group* G *with* $m > 2$*. Let*

$$T_\chi = \left\{ \frac{4a_\chi^2\lambda}{b_\chi^2 - a_\chi^2}, \frac{(b_\chi + a_\chi)\lambda}{b_\chi - a_\chi}, \frac{(b_\chi - a_\chi)\lambda}{b_\chi + a_\chi} \right\} \quad \text{and} \quad U_\chi = \begin{cases} T_\chi & \text{if } |\widehat{G}^0| = 0, \\ T_\chi \cup \{\lambda\} & \text{if } |\widehat{G}^0| > 0. \end{cases} \tag{5.15}$$

For each $u \in U_\chi$*, if* q *is a prime divisor of* u *and* q *is a primitive root mod* p^s*, then* $q^f \mid\mid u$ *for some even* f*.*

Proof. For each $u \in U_\chi$, by (2.2), (2.3) and (5.4) there is a subset E_χ of G for which $|\chi(E_\chi)|^2 = u$. Since $|G| = p^s$, we have $\exp(G) = p^e$ for some integer $e \leq s$ and so $\chi(E_\chi) \in \mathbb{Z}[\zeta_{p^e}]$. Now if q is a primitive root mod p^s, then q is a primitive root mod p^e. Apply Lemma 4.2 with $X = \chi(E_\chi)$. □

We now illustrate the use of Theorem 5.15 to rule out the existence of a $(6561, 42, 120, 90)$-SEDF.

Example 5.16. *Suppose, for a contradiction, that there exists a* $(6561, 42, 120, 90)$*-SEDF. By Lemma 5.1,*

$$(a_\chi, b_\chi) \in \{(1,2),(1,4),(1,5),(4,5)\} \quad \text{for all } \chi \in \widehat{G}^N.$$

Since 5 *is a primitive root mod* 6561*, by Theorem 5.15 we cannot have* $(a_\chi, b_\chi) \in \{(1,2),(1,5),(4,5)\}$ *otherwise the set* T_χ *defined in* (5.15) *contains an element* u *for which* $5 \mid\mid u$*. Therefore* $(a_\chi, b_\chi) = (1,4)$ *for all* $\chi \in \widehat{G}^N$*, and so the SEDF satisfies the simple character value property with respect to* $(1,4)$*. Since* 2 *is a primitive root mod* 6561*, Theorem 5.10 then gives the contradiction that* 4 *is odd.*

Remark 5.17. *Combination of the nonexistence results of Proposition 1.3 (4), Lemmas 1.4, 5.1, and Theorems 4.3, 5.4, 5.12, 5.15, shows that there is no nontrivial (v, m, k, λ)-SEDF for $v \le 10^5$ and $m \in \{5, 6\}$; and that for $v \le 10^4$ and $m > 2$ there are only 70 possible parameter sets for a nontrivial (v, m, k, λ)-SEDF that is not near-complete, namely:*

$\{(540, 12, 42, 36), (784, 30, 18, 12), (1089, 35, 24, 18), (1540, 77, 18, 16), (1701, 35, 30, 18),$
$(1701, 35, 40, 32), (2058, 86, 22, 20), (2376, 11, 190, 152), (2401, 7, 280, 196), (2401, 9, 60, 12),$
$(2401, 9, 120, 48), (2401, 9, 180, 108), (2401, 9, 240, 192), (2401, 16, 120, 90), (2401, 37, 40, 24),$
$(2401, 65, 30, 24), (2500, 18, 105, 75), (2500, 35, 42, 24), (2500, 52, 42, 36), (2601, 53, 40, 32),$
$(2625, 42, 48, 36), (2646, 16, 138, 108), (2784, 116, 22, 20), (3025, 57, 36, 24), (3381, 23, 130, 110),$
$(3888, 24, 156, 144), (3888, 47, 52, 32), (3888, 47, 78, 72), (3969, 32, 112, 98), (4096, 8, 390, 260),$
$(4096, 14, 105, 35), (4096, 14, 210, 140), (4225, 67, 48, 36), (4375, 7, 162, 36), (4375, 7, 324, 144),$
$(4375, 7, 486, 324), (4375, 7, 540, 400), (4375, 9, 405, 300), (4375, 16, 270, 250), (4375, 37, 108, 96),$
$(4375, 37, 54, 24), (4375, 37, 81, 54), (4564, 163, 26, 24), (4625, 37, 68, 36), (5376, 44, 75, 45),$
$(5376, 44, 100, 80), (5776, 78, 60, 48), (5832, 8, 595, 425), (5832, 8, 714, 612), (5832, 18, 147, 63),$
$(5832, 18, 294, 252), (5832, 35, 98, 56), (5832, 86, 49, 35), (5888, 92, 58, 52), (6400, 80, 54, 36),$
$(6656, 26, 121, 55), (6656, 26, 242, 220), (6860, 20, 266, 196), (6860, 58, 95, 75), (6976, 218, 30, 28),$
$(8281, 93, 60, 40), (8625, 23, 140, 50), (8625, 23, 280, 200), (8960, 7, 1054, 744), (8960, 32, 238, 196),$
$(9801, 13, 420, 216), (9801, 26, 308, 242), (9801, 57, 140, 112), (9801, 101, 70, 50), (9801, 101, 84, 72)\}$.

In Section 4 we proved the exponent bound of Theorem 4.3 using only information about the SEDF parameters (v, m, k, λ), and applied it to the case $m = 2$. We now derive a different exponent bound that uses information about the possible values of $|\chi(D)|^2$ and $|\chi(D_1)|^2$, and apply it to two of the open cases with $m > 2$ given in Remark 5.17.

Theorem 5.18. *Suppose $\{D_1, D_2, \ldots, D_m\}$ is a (v, m, k, λ)-SEDF in a group G, let $D = \bigcup_{i=1}^m D_i$, and let p be a prime dividing v. Suppose U is a subgroup of G for which $U \cap G_p = \{1\}$ and p is self-conjugate modulo* $\exp(G/U)$.

(1) If $|\chi(D)|^2 \equiv 0 \pmod{p^{2d}}$ for every nonprincipal $\chi \in \widehat{G}$, then

$$\exp(G_p) \le \max\left\{ \frac{|U|}{p^d}|G_p|, \frac{p\,|\widehat{G}^0| \cdot |U|}{(p-1)v}|G_p| \right\}.$$

(2) If $|\chi(D_1)|^2 \equiv 0 \pmod{p^{2d}}$ for every nonprincipal $\chi \in \widehat{G}$, then $\exp(G_p) \le \dfrac{|U|}{p^d}|G_p|$.

Proof. The proof is analogous to that of [5, Chapter VI, Theorem 15.11]. We prove only (1); the proof of (2) is similar.

Let W be a subgroup of G_p for which G_p/W is cyclic of order $\exp(G_p)$. It follows from $U \cap G_p = \{1\}$ that $U \cap W = \{1\}$, and so we may write $H = U \times W$. Since $\exp(G_p/W) = \exp(G_p)$, we then have $\exp(G/H) = \exp(G/U)$. Let ρ be the canonical epimorphism $\rho : G \to G/H$. Then by assumption,

$$|\widetilde{\chi}(\rho(D))|^2 \equiv 0 \pmod{p^{2d}} \quad \text{for every nonprincipal } \widetilde{\chi} \in \widehat{G/H}.$$

Since p is self-conjugate modulo $\exp(G/H)$, this implies [5, Chapter VI, Lemma 13.2]

$$\widetilde{\chi}(\rho(D)) \equiv 0 \pmod{p^{d}} \quad \text{for every nonprincipal } \widetilde{\chi} \in \widehat{G/H},$$

and then by Ma's Lemma [19] we have

$$\rho(D) = p^d X_0 + P X_1,$$

where $X_0, X_1 \in \mathbb{Z}[G/H]$ have non-negative coefficients and P is the unique subgroup of G/H of order p.

In the case $X_0 \neq 0$, we have $p^d \leq |H| = |U| \cdot |W| = |U| \cdot \frac{|G_p|}{\exp(G_p)}$, which rearranges to $\exp(G_p) \leq \frac{|U|}{p^d}|G_p|$.

Otherwise, in the case $X_0 = 0$, we have $\rho(D) = PX_1$. Now consider the $\frac{|G|}{|H|}(1-\frac{1}{p})$ characters $\widetilde{\chi} \in \widehat{G/H}$ which are nonprincipal on P. Each such character satisfies $\widetilde{\chi}(\rho(D)) = 0$, and its associated lifting character $\chi \in \widehat{G}$ satisfies $\chi(D) = 0$. Therefore by the definition (2.3) of $\widehat{G^0}$, we have $|\widehat{G^0}| \geq \frac{|G|}{|H|}(1-\frac{1}{p})$, which implies $\exp(G_p) \leq \frac{p\,|\widehat{G^0}|\cdot|U|}{(p-1)v}|G_p|$. □

We now illustrate the use of Theorem 5.18 to obtain an exponent bound on a group containing a $(2401, 7, 280, 196)$-SEDF and a group containing a $(5832, 8, 595, 425)$-SEDF.

Example 5.19. *Suppose there exists a* $(2401, 7, 280, 196)$*-SEDF in a group* G*. Note that* $196 = 2^2 \cdot 7^2$ *and that neither* 2 *nor* 7 *is a primitive root mod* $7^4 = 2401$*, so Theorem 4.3 does not apply. However, by Lemma 5.1 the SEDF satisfies the simple character value property with respect to* $(a, b) = (3, 5)$*, and so from Table 5.1 we have* $|\chi(D_1)|^2 \in \{7^2, 4\cdot 7^2, 16\cdot 7^2\}$ *for every nonprincipal* $\chi \in \widehat{G}$*. Since* 7 *is self-conjugate modulo* 2401*, we may apply Theorem 5.18 (2) with* $(p, d) = (7, 1)$ *and* $U = \{1\}$ *to show that* $\exp(G) \leq 7^3$*.*

Example 5.20. *Suppose there exists a* $(5832, 8, 595, 425)$*-SEDF in* G*. Note that* $5832 = 2^3 \cdot 3^6$ *and* $425 = 5^2 \cdot 17$*. Theorem 4.3 does not give any constraint on the structure of* G *(even though it may be applied with* $(p, q) = (3, 5)$*). By Lemma 5.1, the SEDF satisfies the simple character value property with respect to* $(2, 3)$*, and so from Table 5.1 we have* $|\chi(D)|^2 \in \{0, 2^4 \cdot 5 \cdot 17\}$ *and* $|\chi(D_1)|^2 \in \{5 \cdot 17, 5^2 \cdot 17, 5^3 \cdot 17\}$ *for every nonprincipal* $\chi \in \widehat{G}$*, and* $|\widehat{G^0}| = 2079$ *from Theorem 5.4. In this case Theorem 5.18 (2) does not apply. However, because* 2 *is self-conjugate modulo* $2^3 \cdot 3^6$*, we may apply Theorem 5.18 (1) with* $(p, d) = (2, 2)$ *and* $U = \{1\}$ *to obtain* $\exp(G_2) \leq \max\{2, \frac{154}{27}\} < 2^3$*. Therefore* $\exp(G_2) \leq 2^2$*.*

6 Concluding remarks

We have presented a comprehensive treatment of SEDFs, using character theory and algebraic number theory to derive many nonexistence results. We have characterized the parameters of a nontrivial near-complete SEDF, and constructed a $(243, 11, 22, 20)$-SEDF in $\mathbb{Z}_3^5$ from a detailed analysis of the action of the Mathieu group M_{11} on the points of the projective geometry $PG(4, 3)$. This is the first known nontrivial example of SEDF with $m > 2$.

As we were finalizing our paper, Wen, Yang and Feng posted a preprint [26] in which they independently constructed a $(243, 11, 22, 20)$-SEDF in $\mathbb{Z}_3^5$ using cyclotomic classes over $\mathbb{F}_{3^5}$. Their method, which was used shortly afterwards to construct some generalizations of SEDFs [27], is very different from ours.

In closing, we note that until now SEDFs have been considered only in abelian groups. We ask: are there examples of nontrivial SEDFs in nonabelian groups?

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