Wavelength Isolation Sequence Pairs

Jonathan Jedwab ⋆ and Jane Wodlinger ⋆⋆

Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC, V5A 1S6, Canada
{jed, jwodling}@sfu.ca
9 January 2012 (revised 14 March 2012)

Abstract. In the early 1950s, Golay studied binary sequence pairs whose autocorrelation properties are ideal for use in the design of multislit spectrometers. He found examples for some small lengths, but, unable to find more, he suggested that perhaps further examples do not exist and turned his attention to an alternative solution to the design problem (involving what are now called Golay complementary sequence pairs). The original sequence pairs that Golay sought appear to have been overlooked in the sixty years since. We present examples of these pairs, which we call *wavelength isolation sequence pairs*, for two new lengths. We provide structural constraints on these sequence pairs and describe a method whereby each of the currently known examples can be constructed from a perfect Golomb ruler.

1 History and Motivation

A spectrometer is a device that produces a spectrum from a source of electromagnetic radiation (see [3] for background on spectrometers). For example, such a device may be used in the analysis of light emitted from an unknown incandescent material in order to establish its chemical makeup. When the incident radiation comprises more than one wavelength, it is often desirable to distinguish a particular wavelength of interest from background radiation.

In 1951, Golay [5] discussed a spectrometer design that isolates radiation of interest (desired radiation) from background radiation by processing incoming radiation in two 'streams', each consisting of an entrance mask, an exit mask and a detector. The entrance and exit masks are opaque surfaces with a pattern of narrow, equally spaced rectangular slits through which radiation passes on its way to identical detectors. The principle is that if radiation of a background wavelength is always passed through the two streams in equal quantities, while radiation of the desired wavelength is passed differentially by the two streams, then the difference in total energy as measured by the two detectors is wholly attributable to radiation of the desired wavelength. Fig. 1 shows a schematic representation of such a spectrometer.

⋆ J. Jedwab is supported by an NSERC Discovery Grant.
⋆⋆ J. Wodlinger was supported by an NSERC Canada Graduate Scholarship.

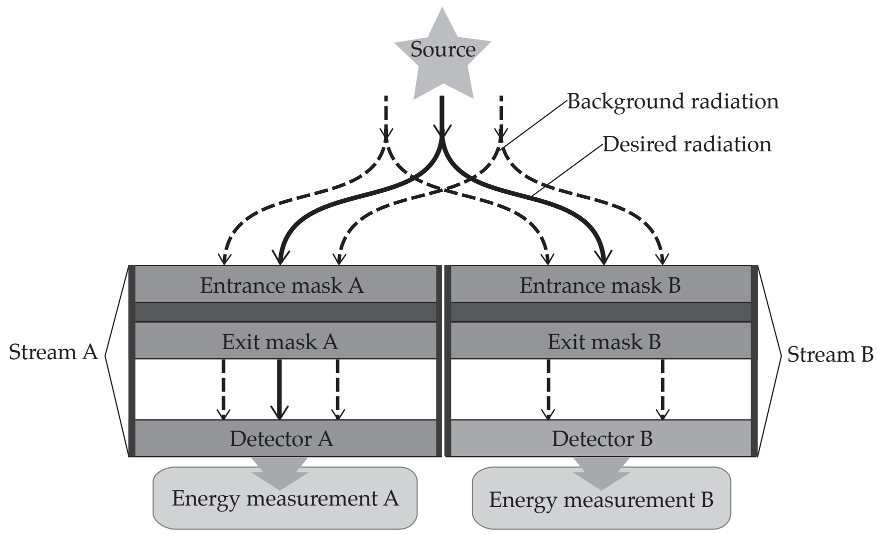


Fig. 1. Schematic representation of Golay's spectrometer design

Golay's mulitislit spectrometer design takes advantage of diffraction to regulate the passage of radiation through the two streams. Diffraction causes radiation to bend as it passes through a narrow opening. A pattern of "open" and "closed" slits is inscribed in the (otherwise opaque) surface of each entrance mask; incident radiation is blocked by the closed slits but passes through the open slits and is diffracted. Since the angle of diffraction varies with wavelength, this separates the incoming radiation into a spectrum, so that each wavelength can be treated differently as it passes through the rest of the spectrometer. In particular, the exit masks are similarly inscribed with a pattern of open and closed slits, which block some radiation and pass the rest to the detectors. The amount of radiation of a given wavelength that is passed by each stream is determined by the entrance and exit slit patterns. The slit patterns must be chosen to isolate the desired wavelength reliably while still being easy to manufacture (thus having relatively few slits).

In the following discussion, we assume that the desired radiation does not undergo diffraction, and thus will reach the detector whenever there is an open slit in the exit pattern aligned with an open slit in the entrance pattern. Then, if background radiation of wavelength λ_v is diffracted such that it arrives at the exit mask v positions (slits) to the right or left, then radiation of wavelength λ_v will reach the detector whenever there is an open slit in the exit mask v positions to the right or left, respectively, of an open slit in the entrance mask. The case where the desired radiation does undergo diffraction can be treated by simply translating both exit masks by an appropriate amount relative to the entrance masks.

Golay represented the entrance and exit slit patterns as binary $\{0,1\}$ sequences, in which 0s represent closed slits and 1s represent open slits. Fig. 2 shows radiation of background wavelength λ_1, which is diffracted by one position to the right, passing through the entrance and exit masks of one stream of a spectrometer, along with the binary sequences associated with the entrance and exit slit patterns. The stream pictured allows one passage of radiation of wavelength λ_1 to the detector.

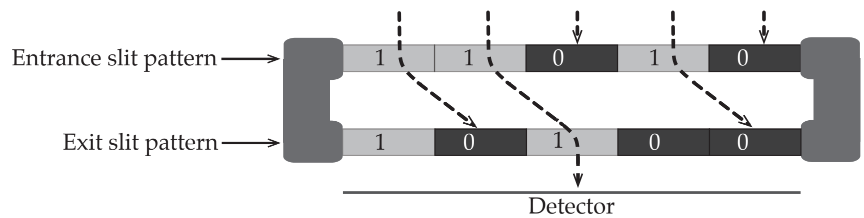


Fig. 2. Example of one stream of a multislit spectrometer

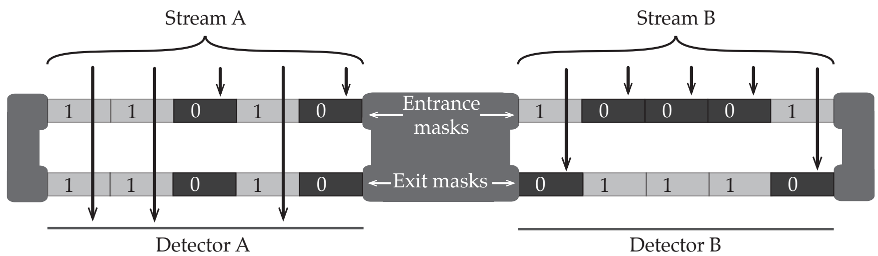
Golay [5] proposed that effective isolation of the desired wavelength could be achieved by entrance slit patterns A and B and exit slit patterns A' and B' with the following properties.

(a) A' is an exact copy of A, and B' is the complement of B.
(b) The number of open slits in A that are followed at distance $v > 0$ (reading from left to right) by an open slit is equal to the number of open slits in B that are followed at distance v by a closed slit, and also equal to the number of closed slits in B followed at distance v by an open slit.

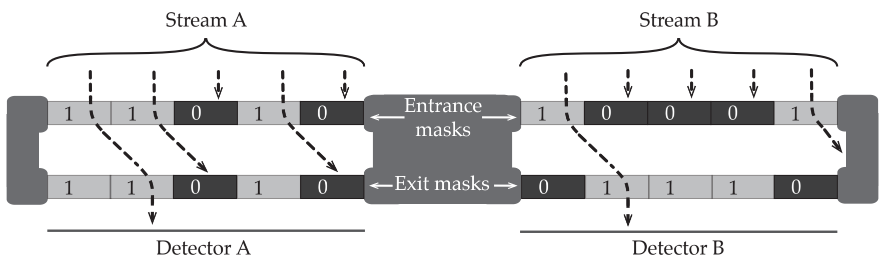
Condition (a) guarantees that all of the desired radiation passed by entrance slit pattern A reaches the detector whereas none of the desired radiation passed by entrance slit pattern B does so. Condition (b) guarantees that radiation of a background wavelength is always passed identically by the two streams, whether it is diffracted to the right (hence the open-closed condition) or the left (hence the closed-open condition).

Since the two exit slit patterns are determined by the two entrance slit patterns, the optical system described above is modeled by an ordered pair of binary $\{0,1\}$ sequences A and B, which represent the entrance slit patterns A and B, respectively. The system illustrated in Fig. 3 corresponds to the sequence pair $A = (11010)$, $B = (10001)$. Fig. 3(a) shows the differential passage of the desired wavelength through both streams, while Fig. 3(b) shows the identical passage of background wavelength λ_1 through both streams.

In 1951, Golay found examples of sequences satisfying Conditions (a) and (b) by hand for lengths 3, 5 and 8 [5]. Unable to find further (nontrivial) examples, he stated that "the possibility must be reckoned with, that solutions for such



(a) Passage of desired radiation through both streams of a multislit spectrometer



(b) Passage of one wavelength of background radiation through both streams of a multislit spectrometer

Fig. 3. Example of a multislit spectrometer with entrance and exit slit patterns satisfying Conditions (a) and (b)

patterns with more than 8 slits do not exist." He diverted his attention to an alternative solution to the problem— one that uses a two-row array of slits rather than a single row, the patterns for which can be constructed for infinitely many lengths using what are now known as Golay complementary sequence pairs (see, for example, [6], [10], [7] for background on these complementary pairs). The search for sequences suitable for single row entrance slit patterns, which we have termed *wavelength isolation sequence pairs (WISPs)*, was apparently forgotten for the next sixty years.

In Sect. 2 we show that in fact there is a WISP of length 13 as well as a WISP of length 7 that Golay overlooked. We then present some structural constraints on WISPs. In Sect. 3 we describe a construction method that explains all of the known examples of WISPs, by making a connection to perfect Golomb rulers. In Sect. 4 we provide partial results on the classification of all WISPs.

2 Structural Constraints on WISPs

Let $A = (a_0, \ldots, a_{t-1})$ be a binary $\{0, 1\}$ sequence of length t and let $x, y \in \{0, 1\}$. For $v \geq 0$, we define

$$S_A(x, y, v) = \left|\{(j, j+v) : (a_j, a_{j+v}) = (x, y) \text{ and } 0 \leq j < t - v\}\right|$$

to be the number of positions in A containing an x followed at distance v by a y. For example, if $A = (10100100)$ then $S_A(1,1,3) = 1$ and $S_A(1,0,4) = 2$. We note that $S_A(1,1,v)$ is the *aperiodic autocorrelation* of the $\{0,1\}$ sequence A. We write $w(A)$ for the weight of A, namely its number of 1s. We now formally define a WISP.

Definition 1. *Let $A = (a_0, \ldots, a_{t-1})$ and $B = (b_0, \ldots, b_{t-1})$ be binary sequences of length t. We say that (A, B) is a* wavelength isolation sequence pair (WISP) *if*

$$w(A) \geq 1 \quad \textit{and} \tag{1}$$

$$S_A(1,1,v) = S_B(1,0,v) = S_B(0,1,v) \quad \textit{for } 1 \leq v < t \ . \tag{2}$$

It is easily verified by reference to Condition (b) in Sect. 1 that if A and B form a WISP then they will be suitable for use as the entrance slit patterns of a multislit spectrometer (Condition (1) ensures that some radiation is passed). Without loss of generality, we can take $a_0 = 1$ (by left-shifting the elements of A and padding with zeroes on the right). We can also form an equivalent WISP by reversing the subsequence of A from its initial '1' element a_0 to its final '1' element. Further, if (A, B) is a WISP then so is $(A, \bar{B})$, where $\bar{B}$ is the complement of B, since $S_B(1,0,v) = S_{\bar{B}}(0,1,v)$. Thus we may take $w(B) \leq \frac{t}{2}$. There is a WISP of every length, namely $A = (10\ldots0)$ and $B = (0\ldots0)$, whose corresponding multislit spectrometer is trivial. We consider a WISP to be nontrivial if $w(A) > 1$. Up to equivalence, there are five known nontrivial examples of WISPs, as presented in Table 1. The examples in the first column were known to Golay [5], while the examples in the second column are new.

Table 1. All known nontrivial WISPs, up to equivalence

$\begin{cases} A = (110) \\ B = (010) \end{cases}$	$\begin{cases} A = (1101000) \\ B = (0001000) \end{cases}$
$\begin{cases} A = (11010) \\ B = (10001) \end{cases}$	$\begin{cases} A = (1100101000000) \\ B = (0000001000000) \end{cases}$
$\begin{cases} A = (11001010) \\ B = (10000001) \end{cases}$	

We now present an important structural constraint on WISPs.

Proposition 2. *If A and B form a WISP then B is symmetric.*

Proof. Suppose that $A = (1, a_1 \ldots a_{t-1})$ and $B = (b_0\, b_1 \ldots b_{t-1})$ form a WISP of length $t > 1$. Then by (2) with $v = t - 1$, we obtain

$$b_0 = b_{t-1} \ . \tag{3}$$

We may therefore take $t > 3$. We now prove by induction on i that $b_i = b_{t-1-i}$ for $0 \le 2i < t-1$, so that B is symmetric. The base case $i = 0$ is given by (3). Assume that cases up to $i-1$ hold, where $2 \le 2i < t-1$, so that B has the form

$$B = (\, b_0\, b_1 \;\cdots\; b_{i-1} \,\big|\, b_i \;\cdots\; b_{t-1-i} \,\big|\, b_{i-1} \;\cdots\; b_1\, b_0 \,) \ .$$

We wish to prove that $b_i = b_{t-1-i}$.

By (2) with $v = t-1-i$, we have

$$S_B(1, 0, t-1-i) = S_B(0, 1, t-1-i) \ . \tag{4}$$

But by the inductive hypothesis, $(b_j, b_{j+t-1-i}) = (b_{t-1-j}, b_{i-j})$ for $1 \le j \le i-1$, so that the contributions to $S_B(1, 0, t-1-i)$ arising from index pairs $(j, j+t-1-i)$ with $1 \le j \le i-1$ are exactly balanced by the contributions to $S_B(0, 1, t-1-i)$ arising from index pairs $(i-j, t-1-j)$ with $1 \le j \le i-1$. Accounting for the remaining contributions to $S_B(1, 0, t-1-i)$ and $S_B(0, 1, t-1-i)$ from index pairs $(0, t-1-i)$ and $(i, t-1)$, and using (4), then gives

$$(b_0, b_{t-1-i}) = (1, 0) \Leftrightarrow (b_i, b_{t-1}) = (0, 1)$$

and

$$(b_0, b_{t-1-i}) = (0, 1) \Leftrightarrow (b_i, b_{t-1}) = (1, 0) \ .$$

Using (3), we obtain $b_i = b_{t-1-i}$ as required, thus completing the induction.

In light of the symmetry of B, the conditions on a WISP may be rephrased to give an alternative definition.

Alternative Definition 3. *Let $A = (a_0, \ldots, a_{t-1})$ and $B = (b_0, \ldots, b_{t-1})$ be binary sequences of length t. We say that A and B form a* wavelength isolation sequence pair (WISP) *if*

$$B \text{ is symmetric,}$$

$$w(A) \ge 1 \quad \text{and} \tag{5}$$

$$S_A(1, 1, v) = S_B(1, 0, v) \quad \text{for } 1 \le v < t \ . \tag{6}$$

We will present a second structural constraint in Proposition 5 concerning the weights of members of WISPs. In preparation, we will prove Lemma 4. For $v \ge 0$, we define

$$P_A(x, y, v) = \left|\{(j, j+v) : (a_j, a_{(j+v) \bmod t}) = (x, y) \text{ and } 0 \le j < t\}\right| \ ,$$

a periodic analogue of $S_A(x, y, v)$.

Lemma 4. *For every binary $\{0,1\}$ sequence C of length t,*

$$\sum_{v=1}^{t-1} P_C(1, 1, v) = w(C)^2 - w(C) \ . \tag{7}$$

Furthermore, if A and B form a WISP of length t, then

$$P_A(1, 1, v) + P_B(1, 1, v) = w(B) \quad \text{for } 1 \le v < t \ . \tag{8}$$

Proof. For (7), we note that

$$\sum_{v=1}^{t-1} P_C(1,1,v) = w(C)(w(C)-1) \ , \tag{9}$$

since each ordered pair of distinct '1' entries in C contributes exactly 1 to the sum. It is easily verified that

$$S_C(x,y,v) + S_C(x,y,t-v) = P_C(x,y,v) \quad \text{ for } 1 \le v < t \tag{10}$$

(which is a restatement of a well-known relation between the periodic and aperiodic autocorrelations of a binary sequence). Let $1 \le v < t$. Applying (10) with $(C,x,y) = (A,1,1)$ and $(B,1,0)$ gives

$$P_A(1,1,v) = P_B(1,0,v) \ , \tag{11}$$

by (6). There are $w(B)$ 1s in B, of which $P_B(1,1,v)$ are followed by a 1 at (periodic) distance v and $P_B(1,0,v)$ are followed by a 0. Therefore

$$P_B(1,0,v) + P_B(1,1,v) = w(B) \ ,$$

which combines with (11) to give (8).

Proposition 5 now follows easily from Lemma 4.

Proposition 5. *Suppose that* $A = (a_0,\ldots,a_{t-1})$ *and* $B = (b_0,\ldots,b_{t-1})$ *form a WISP of length* t*. Then*

$$w(A)^2 + w(B)^2 = w(B)\,t + w(A) \ .$$

Proof. Summing (8) over $v = 1,\ldots,t-1$ gives

$$\sum_{v=1}^{t-1} P_A(1,1,v) + \sum_{v=1}^{t-1} P_B(1,1,v) = (t-1)\,w(B) \ .$$

Substitution from (7) gives the result.

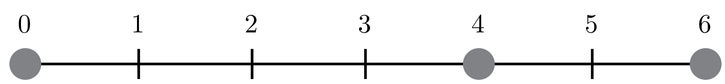
3 Construction of WISPs from Perfect Golomb Rulers

Golomb rulers were studied by Babcock [2] in 1953 for use in eliminating third-order interference between radio communications channels. Since then, they have been studied by different researchers under a variety of names, including *distinct difference sets* [1]. They are named for Prof. Solomon Golomb, who conducted a systematic study of their properties (see [4] for details). In addition to Babcock's application in radio communications, Golomb rulers can be used in X-ray crystallography to distinguish crystal lattice structures whose diffraction patterns are identical; in coding theory to produce self-orthogonal codes; and in radio astronomy, both in locating distant radio sources and in determining the best layout of linear antenna arrays [8].

Golomb rulers are equivalent to Sidon sets [4], as defined by Sidon [9] in 1932 in connection with a problem in combinatorial number theory; the two objects were studied independently for many years before the connection was made [4].

Definition 6. *A* Golomb ruler *is a set of marks at integer positions along a ruler such that no two distinct pairs of marks are the same distance apart. The number n of marks is called the* order *of the ruler and the largest distance ℓ between any two marks is called the* length *of the ruler.*

Example 7. Consider the following ruler of length 6 and order 3, with marks at positions 0, 4 and 6.



The distances between pairs of marks are 2, 4 and 6, so the above is a Golomb ruler.

By convention, we write the set of marks $\{m_0, m_1, \ldots, m_{n-1}\}$ of a Golomb ruler of length ℓ and order n in increasing order, taking the smallest mark to be 0 so that the largest is ℓ.

Definition 8. *A Golomb ruler R of length ℓ and order n is* perfect *if for every integer d satisfying $1 \leq d \leq \ell$, there is exactly one pair of marks $m_1, m_2 \in R$ such that $m_2 - m_1 = d$.*

Clearly, a Golomb ruler of length ℓ and order n satisfies $\ell \geq \binom{n}{2}$; if the ruler is perfect then $\ell = \binom{n}{2}$. A perfect Golomb ruler can be obtained from the Golomb ruler in Example 7 by adding a mark at position 1.

Theorem 9 describes two construction procedures, each of which produces a WISP from a perfect Golomb ruler of length ℓ. The constructed WISPs are inequivalent for $\ell \neq 1$.

Theorem 9. *Let R be a perfect Golomb ruler of order $n \geq 1$ and length $\ell = \binom{n}{2}$. For $0 \leq j \leq \ell$, let*

$$c_j = \begin{cases} 1 & \text{for } j \in R; \\ 0 & \text{otherwise} \end{cases} .$$

Then

$$\begin{cases} A = (1\, c_1 \ldots c_\ell\, 0) \\ B = (1\, 0\, \ldots 0\ 1) \end{cases} \tag{12}$$

is a WISP of length $\ell + 2$ and

$$\begin{cases} A = (c_0 \dots c_{\ell-1}\ 1\ 0 \dots 0) \\ B = (0\ \dots\ 0\ \ 1\ 0 \dots 0) \end{cases} \tag{13}$$

is a WISP of length $2\ell + 1$.

Proof. We will show that A and B satisfy the conditions of Alternative Definition 3 under both Constructions (12) and (13). Clearly, Condition (5) is satisfied and B is symmetric in both cases, so we need only show that Condition (6) is satisfied in both cases (with $t = \ell + 2$ for the pair (12) and $t = 2\ell + 1$ for the pair (13)). By construction, the positions of the 1s in A are the marks of the perfect Golomb ruler R, and

$$S_A(1,1,v) = S_B(1,0,v) = \begin{cases} 1 & \text{for } 1 \le v \le \ell \\ 0 & \text{for } v > \ell\ . \end{cases}$$

There are only four perfect Golomb rulers, the longest of which has length 6.

Proposition 10. (Golomb, see [4]) *Up to reversal and translation, the only perfect Golomb rulers are*

Order (n)	Length (ℓ)	Ruler
1	0	$\{0\}$
2	1	$\{0, 1\}$
3	3	$\{0, 1, 3\}$
4	6	$\{0, 1, 4, 6\}$

Each of the known WISPs (or one that is equivalent), presented in Table 1, can be constructed by (12) or (13) from one of the perfect Golomb rulers listed in Proposition 10. Trivial WISP lengths 2 and 1 arise from the perfect Golomb ruler of length 0, WISP lengths 3 and 3 from length 1, WISP lengths 5 and 7 from length 3, and WISP lengths 8 and 13 from length 6.

4 Are There WISPs of Length Greater Than 13?

There are no more perfect Golomb rulers to use in Theorem 9, and computer search rules out the existence of additional WISPs for lengths less than 32. We were unable to determine whether there are any more WISPs. However, Propositions 11 and 12 give partial results on the classification of all WISPs.

Proposition 11. *Up to equivalence, the only nontrivial WISPs* (A, B) *with* $w(B) = 1$ *are those listed in Table 1.*

Proof. Let A and B form a nontrivial WISP of length t with $w(B) = 1$. Then, since B is symmetric, t is odd and

$$b_i = \begin{cases} 1 & \text{for } i = \frac{t-1}{2} \\ 0 & \text{otherwise}\ . \end{cases}$$

Thus

$$S_B(1,0,v) = \begin{cases} 1 & \text{for } 1 \le v \le \frac{t-1}{2} \\ 0 & \text{for } \frac{t-1}{2} < v < t \end{cases} ,$$

which, since A and B form a WISP, forces

$$S_A(1,1,v) = \begin{cases} 1 & \text{for } 1 \le v \le \frac{t-1}{2} \\ 0 & \text{for } \frac{t-1}{2} < v < t \end{cases} .$$

Then the subsequence of A from its initial to final '1' element (this sequence having $\frac{t-1}{2}+1$ elements) is a perfect Golomb ruler of length $\frac{t-1}{2}$. By Proposition 10, $\frac{t-1}{2} = 0, 1, 3$ or 6 and A is determined up to reversal and translation.

A similar argument shows that WISPs with $B = (10\ldots01)$ are characterised by Construction (12) of Theorem 9. Proposition 12 rules out another case in which $w(B) = 2$, where the 1s are in the central positions of B.

Proposition 12. *Suppose that A and B form a WISP of length $t > 2$. Then $B \neq (0\ldots0110\ldots0)$.*

Proof. Suppose for a contradiction that $B = (0\ldots0110\ldots0)$. Then t is even, and

$$S_A(1,1,v) = S_B(1,0,v) = \begin{cases} 1 & \text{for } v = 1, \frac{t}{2} \\ 2 & \text{for } 2 \le v \le \frac{t}{2} - 1 \\ 0 & \text{for } \frac{t}{2} < v < t \end{cases} . \tag{14}$$

Without loss of generality, applying (14) $\frac{t}{2}$ times with $v = t-1, t-2, \ldots, \frac{t}{2}$, respectively, gives $A = (1a_1 \ldots a_{\frac{t}{2}-1}10\ldots0)$. Then in the case $t = 4$ we derive a contradiction from $S_A(1,1,1) = 1$, and in the case $t > 4$ the condition $S_A(1,1,\frac{t}{2}-1) = 2$ forces $A = (11a_2 \ldots a_{\frac{t}{2}-2}110\ldots0)$, contradicting $S_A(1,1,1) = 1$.

References

1. Atkinson, M., Santoro, N., Urrutia, J.: Integer Sets with Distinct Sums and Differences and Carrier Frequency Assignments for Nonlinear Repeaters. IEEE Trans. Comm. 34, 614–617 (1986)
2. Babcock, W. C.: Intermodulation Interference in Radio Systems. Bell Systems Technical Journal, 63–73 (1953)
3. Busch, K. W., Busch, M. A.: Multielement Detection Systems for Spectrochemical Analysis. Wiley, New York, 147–163 (1990)
4. Dimitromanolakis, A.: Analysis of the Golomb Ruler and the Sidon Set Problems, and Determination of Large, Near-optimal Golomb Rulers. Technical University of Crete, Department of Electronic and Computer Engineering (2002)
5. Golay, M. J. E.: Static Multislit Spectrometry and Its Application to the Panoramic Display of Infrared Spectra. J. Opt. Soc. Am. 41, 468–472 (1951)
6. Golay, M. J. E.: Complementary Series. IRE Trans. Inform. Theory, IT-7, 82–87 (1961)

7. Fiedler, F., Jedwab, J., Parker, M. G.: A Multi-dimensional Approach to the Construction and Enumeration of Golay Complementary Sequences. J. Comb. Theory A, 115, 753–776 (2008)
8. Rankin, W. T.: Optimal Golomb Rulers: An Exhaustive Parallel Search Implementation. Master's Thesis, Duke University, Department of Electrical Engineering (1993)
9. Sidon, S.: Ein Satz Über Trigonometrische Polynome und Seine Anwendungen in der Theorie der Fourier-Reihen. Mathematische Annalen, 106, 536–539 (1932)
10. Turyn, R. J.: Hadamard Matrices, Baumert-Hall Units, Four-Symbol Sequences, Pulse Compression, and Surface Wave Encodings. J. Comb. Theory A, 16, 313–333 (1974)