

Minimum Deficiency of Even Order Permutation Arrays

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1 Introduction

Definition 1.1. *An $n \times n$ permutation array has entries from the set $\{0, 1\}$ such that there is exactly one non-zero entry per column and per row of the array.*

We will index the array following the traditional indexing of matrices (i.e. position $(1, 1)$ refers to the top left entry of the array).

We define a toroidal vector (w, h) between each pair of distinct non-zero entries in the permutation array. The entry w represents movement to the right, and the entry h represents movement downwards. The toroidal vector may cycle around the array as if the array were wrapped completely around a torus, so that (w, h) belongs to the set $[n - 1]^2$. The formal definition of a toroidal vector between two non-zero entries is as follows:

Definition 1.2. *Let $A = (A_{i,j})$ be a permutation array of order n . The toroidal vector from the non-zero entry $A_{i,j}$ to the non-zero entry $A_{k,l}$ is $((l - j) \bmod n, (k - i) \bmod n)$.*

Note that, as $|[n - 1]^2| = (n - 1)^2$, there are $(n - 1)^2$ possible distinct toroidal vectors in a permutation array of order n , and as there is a toroidal vector defined from every non-zero entry to every other non-zero entry, the permutation array contains $n(n - 1)$ toroidal vectors.

Definition 1.3. *Let A be a permutation array of order n . The deficiency of A , denoted $D(A)$, is the number of possible toroidal vectors not contained in A .*

The following theorem establishes a lower bound on the deficiency of a permutation array of a given order.

Theorem 1.4. *([3, Theorem 6], [4, Theorem 1]). Let A be a permutation array of order n . Then*

$$D(A) \geq \begin{cases} n - 1 & \text{for } n \text{ odd} \\ n - 3 & \text{for } n \text{ even.} \end{cases}$$

Definition 1.5. *Let $D_n = \min\{D(A) : A \text{ is an order } n \text{ permutation array}\}$.*

We are interested in determining D_n for certain values of n , when n is even; in particular, we are interested in when the lower bound in Theorem 1.4 is achieved.

Before introducing a construction of permutation arrays that achieve minimum possible deficiency, we give some background on Costas arrays. A Costas array is a type of permutation array. A vector between two non-zero entries in a Costas array of order n is considered to be aperiodic, having positive or negative direction both horizontally and vertically. The aperiodic vector belongs to the set $\{-(n - 1), \dots, (n - 1)\}^2$.

Definition 1.6. *A permutation array A of order n is a Costas array if no two aperiodic vectors formed by joining pairs of distinct non-zero entries are the same.*

Consider the following construction of a Costas array:

Theorem 1.7. (Golomb construction $G_2(q, \varphi, \rho)$ [1]). Let φ and ρ be primitive elements of \mathbb{F}_q , where $q = p^r$ for p prime. Then the permutation array $A = (A_{i,j})$ of order $q - 2$ for which

$$A_{i,j} = 1 \text{ if and only if } \varphi^i + \rho^j = 1$$

is a Costas array.

Definition 1.8. Let G be a Golomb Costas array of order $q - 2$. The augmented Golomb Costas array G_{\pm}^+ associated with G is the $(q - 1) \times (q - 1)$ array formed by augmenting a row of zeros to the lowermost end of G and a column of zeros to the rightmost end of G . The extended Golomb Costas array G' associated with G is formed by setting the $(q - 1, q - 1)$ entry of G_{\pm}^+ to 1.

Theorem 1.9. Let $q = p^r$ where p is prime, let φ and ρ be primitive elements of \mathbb{F}_q , and let G' be the extended Golomb Costas array of order $q - 1$ associated with the $G_2(q, \varphi, \rho)$ Golomb Costas array. Then $D(G') = q - \min\{p, 4\}$.

Starting from a Golomb Costas array G of order $q - 2$, it is sometimes the case that some cyclic permutation of the rows and columns of the extended Golomb Costas array G' yields another Costas array R , having order $q - 1$; in this case R is called a Golomb-Rickard Costas array. Theorem 1.9 was proven for Golomb-Rickard Costas arrays as Theorem 3.3 of [2]; the central focus of that paper is the deficiency of Costas arrays. However, the proof of Theorem 3.3 of [2] applies without modification to the extended Golomb Costas array G' , regardless of whether G' gives rise to a Golomb-Rickard Costas array. The case of $\varphi = \rho$ of Theorem 1.9 was also proven separately in Theorem 14 of [3], without explicit reference to Costas arrays.

Theorem 1.9 implies that, when n is one less than a power of a prime $p > 3$, $D_n = n - 3$. The following table lists D_n for small values of even n , as found by exhaustive search.

n	D_n
4	1
6	3
8	6
10	7
12	9
14	25
16	13
18	15

Table 1: Exhaustive search results

Note that for the values of n in the table above, $D_n = n - 3$ if $n \equiv 0, 4 \pmod{6}$, and $D_n > n - 3$ if $n \equiv 2 \pmod{6}$. This observation, as well as patterns outlined in Section 2 that we noticed in all permutation arrays of order $n \in \{4, 6, 10, 12, 16, 18\}$ achieving deficiency $n - 3$ and all symmetric permutation arrays of order $n \in \{22, 24, 28, 30\}$ achieving deficiency $n - 3$, led to the following conjecture:

Conjecture 1.10. $D_n = n - 3$ if and only if $n \equiv 0, 4 \pmod{6}$.

However, upon realizing that the observations motivating the above conjecture could all be explained by constructions of Golomb Costas arrays, we considered amending our conjecture. After our discovery and exploration of the gap structure (outlined in Section 4) and a new data structure, the graph of repeats, (outlined in Section 5), we amended our conjecture to:

Conjecture 1.11. $D_n = n - 3$ if and only if $n = p^r - 1$, where $p > 3$ is prime.

2 Motivation Behind Conjecture 1.10: The T-Matrix

Definition 2.1. *The T-matrix T of a permutation array A of order n is the $(n-1) \times (n-1)$ matrix whose (w, h) entry is the number of times the toroidal vector (w, h) occurs in A .*

$T_{w,h}$ can be interpreted as the periodic autocorrelation of the array at shift (h, w) . Jedwab and Wodlinger used the concept of the T-matrix in their reformulation [2, Theorem 1.7] of the proof of Theorem 1.4.

Let A be a permutation array of even order n and let T be its T-matrix, and consider the following properties of the T-matrix:

- (i) $D(A)$ is the number of zero entries in T .
- (ii) If the toroidal vector (w, h) joins $A_{i,j}$ to $A_{k,l}$, then the toroidal vector $(n-w, n-h)$ joins $A_{k,l}$ to $A_{i,j}$. Thus toroidal vectors occur in pairs (w, h) and $(n-w, n-h)$, and $T_{w,h} = T_{n-w, n-h}$.
- (iii) $T_{\frac{n}{2}, \frac{n}{2}}$ must be even, by property (ii).
- (iv) Each row sum of T is n , as there are n toroidal vectors of a given width in A . Similarly, each column sum of T is n .
- (v) The path formed by following the toroidal vectors of a given width w starting at non-zero entry $A_{i,j}$ must cycle back to entry $A_{i,j}$. Thus the heights of every toroidal vector of width w sum to zero modulo n , giving the following equation for every row w of T :

$$\sum_{h=1}^{n-1} hT_{w,h} \equiv 0 \pmod{n}.$$

Similarly, for every column h of T , we have:

$$\sum_{w=1}^{n-1} wT_{w,h} \equiv 0 \pmod{n}.$$

- (vi) If row w of T contains no zero entry, then the set of elements in row w is $\{1, 1, \dots, 1, 2\}$ by property (iv). In this case, let h be such that $T_{w,h} = 2$. By property (v), we have

$$\frac{n(n-1)}{2} + h \equiv 0 \pmod{n}$$

and as n is even, $\frac{n(n-1)}{2}$ is an odd multiple of $\frac{n}{2}$, so

$$h \equiv \frac{n}{2} \pmod{n}.$$

Since $1 \leq h \leq n-1$, we have $h = \frac{n}{2}$ and therefore $T_{w, \frac{n}{2}} = 2$. Similarly, if column h of T contains no zero entry, then the set of elements in column h is $\{1, 1, \dots, 1, 2\}$ with $T_{\frac{n}{2}, h} = 2$.

Next we will describe in more detail some additional characteristics of the T-matrix of A assuming $D(A)$ achieves the lower bound $n-3$ of Theorem 1.4.

First note that, as n is even, the deficiency $n-3$ is odd, and so $T_{\frac{n}{2}, \frac{n}{2}} = 0$ (since, by property (ii) above, all zero entries of T at positions other than $(\frac{n}{2}, \frac{n}{2})$ occur in pairs). Furthermore, T contains at most one zero entry per row and per column, which can be shown by considering the case of equality throughout the proof of Theorem 1.4 as given in [2, Theorem 1.7]. Since $D(A) = n-3$

and T is $(n-1) \times (n-1)$, there are $n-3$ rows and $n-3$ columns of T containing exactly one zero entry, and two rows and two columns with no zero entry. These two rows and two columns take the form explained in property (vi) above. By property (ii), these two rows are w and $n-w$ for some w , and likewise for the columns.

Next we describe the rows and columns of T that contain exactly one zero entry. Since there are $n-1$ entries in each row and column of T , and these entries must sum to n , the set of entries in such a row or column must be either

$$\{0, 1, 1, \dots, 1, 2, 2\} \quad (*)$$

or

$$\{0, 1, 1, \dots, 1, 3\} \quad (**).$$

Suppose row w of T has the set of entries given by $(*)$. Let $T_{w,h_0} = 0$, $T_{w,h_2} = 2$, and $T_{w,h'_2} = 2$. Then by property (v) of the T -matrix, we have

$$\frac{n}{2}(n-1) + h_2 + h'_2 - h_0 \equiv 0 \pmod{n}.$$

This equation simplifies to

$$h_2 + h'_2 - h_0 \equiv \frac{n}{2} \pmod{n}.$$

As $1 \leq h_0, h_2, h'_2 \leq n-1$, we have $h_2 + h'_2 - h_0 \in \{-\frac{n}{2}, \frac{n}{2}, \frac{3n}{2}\}$. This congruence also holds for columns of T with set of entries $(*)$.

Suppose row w has the set of entries given by $(**)$. Let $T_{w,h_0} = 0$ and $T_{w,h_3} = 3$. We now have

$$\frac{n}{2}(n-1) + 2h_3 - h_0 \equiv 0 \pmod{n}$$

which simplifies to

$$2h_3 - h_0 \equiv \frac{n}{2} \pmod{n}.$$

Similarly as above, $2h_3 - h_0 \in \{-\frac{n}{2}, \frac{n}{2}, \frac{3n}{2}\}$, and a similar congruence holds for columns of T with set of entries $(**)$.

Now we will demonstrate a few examples of T -matrices of permutation arrays with minimum deficiency.

Consider the permutation $\alpha = [1, 2, 4, 9, 6, 10, 5, 8, 7, 3]$ of order 10 whose corresponding array $(A_{i,j})$ where $\alpha(j) = i$ is shown in Figure 1.

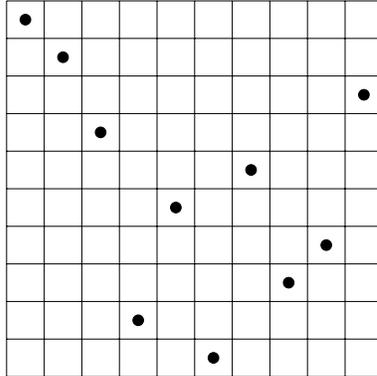


Figure 1: Permutation array $[1, 2, 4, 9, 6, 10, 5, 8, 7, 3]$

The T-matrix corresponding to this permutation array is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & \mathbf{2} & 1 & 1 & 1 & 0 & 1 & 1 & \mathbf{2} \\ 1 & 1 & 1 & 1 & 1 & \mathbf{2} & 1 & \mathbf{2} & 0 \\ \mathbf{2} & 0 & 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 0 & \mathbf{2} \\ 0 & \mathbf{2} & 1 & \mathbf{2} & 1 & 1 & 1 & 1 & 1 \\ \mathbf{2} & 1 & 1 & 0 & 1 & 1 & 1 & \mathbf{2} & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

There are multiple interesting observations to make about the structure of this T-matrix. First, there are no 3s present in the T-matrix. In fact, no 3s are observed in all examples of T-matrices of permutation arrays of order $n \in \{4, 10, 16\}$ whose deficiency is $n - 3$, nor in all examples of T-matrices of symmetric permutation arrays of order $n \in \{22, 28\}$ whose deficiency is $n - 3$. Note that these values of n satisfy $n \equiv 4 \pmod{6}$.

Cycles are formed among the 2s in the T-matrix that are not in the center row and not in the center column; pick one 2 in the T-matrix, say the red 2 at position (2, 2). Find the other 2 in this row, which is at position (2, 9). Looking at the column of this new 2, we find another 2 at position (6, 9). Repeating this procedure, we visit the 2 at position (6, 4), then we find the 2 at position (7, 4), and then the 2 at position (7, 2). We find the other 2 in this column at position (2, 2), the original 2 that we started with. Thus the 2s in rows 2, 6, and 7, or alternatively, the 2s in columns 2, 4, and 9, form a cycle involving three rows and three columns. The blue 2s in this T-matrix also form a cycle among rows 3, 4, and 8 and columns 1, 6, and 8. Note that one of these two cycles is the image of the other under rotational symmetry of the T-matrix (property (ii)). The black 2s, which are in either the center row or center column, are not part of any such cycle.

We can also find a pattern among the positions of the zeros in the T-matrix. We note that $T_{i,(3i) \bmod 10} = 0$ for $i \in [9] \setminus \{1, 9\}$ (excluding rows 1 and 9 as they contain no zero entry). We can also view the pattern of zeros column by column, and note that $T_{(7j) \bmod 10, j} = 0$ for $j \in [9] \setminus \{3, 7\}$ (excluding columns 3 and 7 which contain no zero entry).

We now look at a T-matrix of a permutation array of order 12, where similar observations can be made. Consider the permutation $[1, 3, 2, 11, 7, 10, 5, 8, 12, 6, 4, 9]$ whose array is symmetric about its main diagonal:

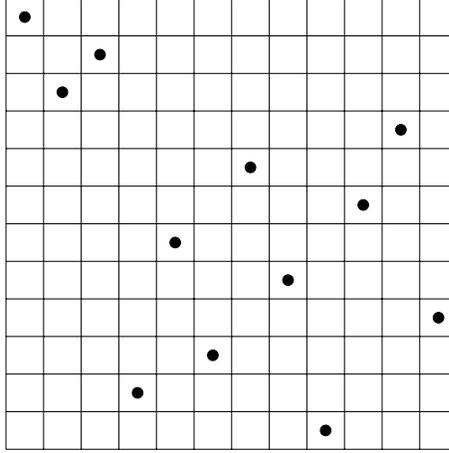


Figure 2: Permutation array $[1, 3, 2, 11, 7, 10, 5, 8, 12, 6, 4, 9]$

Its corresponding T-matrix is notable for being symmetric about its main diagonal and anti-diagonal. Additionally, all zeros are contained in its main diagonal.

$$\begin{pmatrix} 0 & 1 & \mathbf{2} & \mathbf{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{3} & 1 \\ \mathbf{2} & 1 & 0 & 1 & 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 1 \\ \mathbf{2} & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \mathbf{2} & 1 \\ 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 1 & 0 & 1 & \mathbf{2} & 1 \\ 1 & \mathbf{3} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \mathbf{2} & \mathbf{2} & 1 & 0 & 1 \end{pmatrix}$$

We observe the cycles of 2s and regular spacing of zeros, just as in the previous T-matrix. A new feature in this T-matrix is the inclusion of two 3s. We observe exactly two 3s in all examples of T-matrices of permutation arrays of order $n \in \{6, 12, 18\}$ whose deficiency is $n - 3$, and in all examples of T-matrices of symmetric permutation arrays of order $n \in \{24, 30\}$ whose deficiency is $n - 3$. Note that in these cases, $n \equiv 0 \pmod{6}$.

Let A be a permutation array of order $n \in \{4, 6, 10, 12, 16, 18\}$ or a symmetric permutation array of order $n \in \{22, 24, 28, 30\}$, such that $D(A) = n - 3$. Let T be the T-matrix of A .

Observation 2.2. *The entries of 2s in T (not in the center row or column of T) occur in cycles involving exactly three columns and three rows.*

We also observe that the cycles of 2s occur in rotationally symmetric pairs.

Observation 2.3. *There is a $k \in \mathbb{N}$ such that $(k, n) = 1$ and $T_{i, (ik) \bmod n} = 0$ for $i \in [n - 1] \setminus \{w, n - w\}$ where row w of T contains no zero entry.*

There is an equivalent observation with respect to columns.

Observation 2.4.

$$\text{The number of 3s present in } T \text{ is } \begin{cases} 0 & \text{for } n \equiv 4 \pmod{6} \\ 2 & \text{for } n \equiv 0 \pmod{6}. \end{cases}$$

If $n \equiv 0 \pmod{6}$, then either $T_{\frac{n}{6}, \frac{5n}{6}} = T_{\frac{5n}{6}, \frac{n}{6}} = 3$, or $T_{\frac{n}{6}, \frac{n}{6}} = T_{\frac{5n}{6}, \frac{5n}{6}} = 3$.

Observation 2.5. *For all $n \in \{4, 6, 10, 12, 16, 18, 22, 24, 28, 30\}$, there is a symmetric permutation array of deficiency $n - 3$ with corresponding T -matrix that is symmetric about its main diagonal and anti-diagonal, and whose zero entries are contained along its main diagonal.*

Since the cycles of $2s$ occur in pairs, the number of rows of T with set of elements $\{0, 1, 1, \dots, 1, 2, 2\}$ is congruent to 0 modulo 6, similar to how the number of rows of T with set of elements $\{0, 1, 1, \dots, 1, 3\}$ depends on the order of the permutation array modulo 6. These observations, along with the computational results of Table 1, where the n for which $D_n = n - 3$ are such that $n \equiv 0, 4 \pmod{6}$, led to Conjecture 1.10.

However, exhaustive search revealed that all our examples of order n permutation arrays achieving deficiency $n - 3$ occur when $n = p^r - 1$ for $p > 3$ prime, and arise due to constructions of Golomb Costas arrays. In fact, the above observations are consequences of such constructions. Before explaining how these observations can be derived, we present the following definition:

Definition 2.6. *Let $A = (A_{i,j})$ be a permutation array of order n and let $k \in \mathbb{N}$ satisfy $(k, n) = 1$. The k -decimation of A with respect to rows is the array $(A_{((ik-1) \bmod n)+1,j})$.*

The i^{th} row of the k -decimation of A is the $((ki) \bmod n)^{\text{th}}$ row of A , where $i \in [n - 1]$, and the n^{th} row of the k -decimation of A is the n^{th} row of A .

Remark 2.7. *The toroidal vector (w, h) is contained in the k -decimation of A with respect to rows exactly when the toroidal vector $(w, (hk) \bmod n)$ is contained in A .*

Let G be the $G_2(q, \varphi, \rho)$ Golomb Costas array where $q = p^r$ for $p \neq 2$ prime. Let G_{\pm}^+ be the augmented Golomb Costas array of order $n = q - 1$ obtained from G . Let T be the T -matrix of G_{\pm}^+ . If $\varphi^h \neq \rho^w$, then $T_{w,h} = 1$, and otherwise, $T_{w,h} = 0$ [2, Proposition 3.2]. Write $\varphi^k = \rho$, where $(k, n) = 1$ since ρ and φ are both primitive. Then $T_{1,k} = 0$. Indeed, for any $i \in [n - 1]$, $\varphi^{ik} = \rho^i$ and $T_{i,(ik) \bmod n} = 0$. Thus T contains $n - 1$ zero entries and all other entries of T are 1.

Set the (n, n) entry of G_{\pm}^+ to 1 to obtain its extended Golomb Costas array G' . Our objective is to determine when $T_{i,(ik) \bmod n} = 0$, after taking into account the new non-zero entry of G' . There is a toroidal vector of the form $(i, (ik) \bmod n)$ in G' from the new non-zero entry $G'_{n,n}$ to another non-zero entry $G'_{i,j}$ if and only $(i, j) = ((wk) \bmod n, w)$ for some w , because the toroidal vector from $G'_{n,n}$ to $G'_{i,j}$ is (j, i) . Recall that $G'_{i,j} = 1$ and $(i, j) \neq (n, n)$ if and only if $\varphi^i + \rho^j = 1$ (Theorem 1.7). Note that $\varphi^{ki} + \rho^i = 1$ implies $\rho^i + \rho^i = 1$, as $\varphi^k = \rho$. Let $i = w$ be the unique solution to $2\rho^i = 1$ in \mathbb{F}_q . Then for this w , $G'_{(wk) \bmod n, w} = 1$, and the toroidal vector from $G'_{n,n}$ to $G'_{(wk) \bmod n, w}$ is $(w, (wk) \bmod n)$. Furthermore, $T_{w,(wk) \bmod n} = 1$ and $T_{n-w, n-(wk) \bmod n} = T_{n-w, ((n-w)k) \bmod n} = 1$. Since this w is unique, $T_{i,(ik) \bmod n} = 0$ for all $i \in [n - 1] \setminus \{w, n - w\}$, as stated in Observation 2.3. Note that $w \neq n - w$ if and only if $p > 3$ as shown in the proof of Theorem 3.3 of [2]. Thus there are $n - 3$ zero entries in T when $p > 3$.

Next, k -decimate the rows of G' so that G' now satisfies $G'_{i,j} = 1$ if and only if $\varphi^{ki} + \rho^j = \rho^i + \rho^j = 1$, or $(i, j) = (n, n)$. Thus the k -decimation of G' is symmetric about its main diagonal, and so its T -matrix is also symmetric about its main diagonal (as, if $G'_{i,j} = 1$ and $G'_{k,l} = 1$, then $G'_{j,i} = 1$ and $G'_{l,k} = 1$, and toroidal vectors $((l - j) \bmod n, (k - i) \bmod n)$ and $((k - i) \bmod n, (l - j) \bmod n)$ are contained in G'). By Remark 2.7, its T -matrix T' satisfies $T'_{i,i} = 0$ for all $i \in [n - 1] \setminus \{w, n - w\}$ where row w contains no zero entry. After constraining the T -matrix to be symmetric with all zero entries on its main diagonal, Observations 2.2 and 2.4 can be easily derived.

3 Search Algorithm

An exhaustive search of all permutation arrays of order n requires $O(n!)$ time. We used a recursive algorithm to find all permutation arrays of order n having deficiency no greater than a target d , with some strategies to reduce running time in order to search higher values of n .

First, since cyclic rotation of rows and columns preserves the deficiency of a permutation array A , we can assume that $A_{1,1} = 1$. Recall that the total number of toroidal vectors in a permutation array A of order n , counting multiplicity, is $n(n-1)$, and that there are $(n-1)^2$ possible toroidal vectors. Note that there are $(n-1)^2 - D(A)$ distinct toroidal vectors contained in A . Hence the number of repeated toroidal vectors, including multiplicity, is

$$n(n-1) - ((n-1)^2 - D(A)) = (n-1) + D(A).$$

Our algorithm recursively assigns each non-zero entry in A and updates the T-matrix T of A by including the toroidal vectors that are introduced in A by this new entry (i.e. the non-zero entry in column i is assigned, and the toroidal vectors between this entry and each of the non-zero entries in columns 1 through $i-1$ are determined). The algorithm maintains a count of the number of repeated toroidal vectors in A , including multiplicity. The branch of recursion terminates if the count of repeated toroidal vectors exceeds $(n-1) + d$, where d is the target maximum deficiency.

Note, if $d = n - 3$, then T must be of a specific form outlined in Section 2 (the set of entries of each row and column of T must be $\{1, 1, \dots, 1, 2\}$, $\{0, 1, 1, \dots, 1, 2, 2\}$, or $\{0, 1, 1, \dots, 1, 3\}$) so that the number of repeats in each row and column is at most two. If, after updating the T-matrix, the number of repeats in a row or column of T exceeds two, then the branch of recursion is terminated.

Using this algorithm, we can determine D_n for small values of n as shown in Table 1.

As stated in Observation 2.5, for the values of n in Table 1 for which $D_n = n - 3$, we have an example of a symmetric permutation array achieving minimum deficiency. We modified our program to search only symmetric permutation arrays in order to reach larger values of n . The following table highlights some results of this program:

n	D_n
20	?
22	19
24	21
26	23 or 24
28	25
30	27
32	?
34	?

Table 2: Symmetric search results

For $n = 26$, we found a symmetric permutation array with deficiency 24, but not one of deficiency 23. We currently have a program running to search symmetric permutation arrays of order 34. No permutation array with deficiency 31 has been found yet. We are particularly interested in knowing what D_{34} is, because it is the smallest value of n for which our original conjecture (Conjecture 1.10) and our amended conjecture (Conjecture 1.11) give different answers as to whether or not $D_n = n - 3$.

4 Gap Structure

After realizing our initial observations are explained by constructions arising from Golomb Costas arrays, we sought a new perspective on the deficiency of permutation arrays in the hope that this would better our understanding of the problem of minimum deficiency. We used our search algorithm described in Section 3 not only to determine minimum deficiency but also to determine the next smallest attainable deficiencies for small values of n . The following table outlines the results:

n	Attainable deficiency
4	1, 6
6	3, 4, ...
8	6, 9, 10, ...
10	7, 12, 14, 15, ...
12	9, 17, 19, 20, ...
14	25, 26, ...
16	13, 31, 33, 34, ...

Table 3: Exhaustive search results

These computational results suggest that when the lower bound of $n - 3$ for the deficiency can be achieved, there is a large difference between the minimum deficiency and the next smallest attainable deficiency. Alternatively, even if minimum deficiency achieves the lower bound $n - 3$, the point at which a sequence of consecutive deficiency values is attainable is much greater than the lower bound. We refer to these findings as the gap structure.

If we were to consider only those permutation arrays not arising from the Golomb Costas construction, we would find that the minimum deficiency of such permutation arrays for small values of n is much larger than the lower bound $n - 3$. This discovery first gave us the idea that we should replace Conjecture 1.10 with Conjecture 1.11.

We were not able to gain much insight into the gap structure from our original data structure, the T-matrix. Table 3 illustrates that deficiency 8 is not attainable over permutation arrays of order 10. However, here is an example of a 9×9 T-matrix satisfying properties (i) through (vi) of Section 2 with 8 zero entries:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This T-matrix can be generalized to any odd order $n - 1$ by alternating 2s and zeros in the center row and column and placing a 2 in the intersections of rows w and $n - w$ and columns w and $n - w$ provided that the center elements of these rows and columns are zero. Such a T-matrix has $\frac{n-2}{2}$ zeros in its center row and $\frac{n-2}{2}$ zeros in its center column, totalling $n - 2$ zeros (since the element at the intersection of the center row and center column is non-zero).

In order to better our understanding of the gap structure, we came up with a new data structure, the graph of repeats, explained in Section 5.

5 New Data Structure: Graph of Repeats

Let permutation array $A = (A_{i,j})$ of even order n correspond to permutation α for which $\alpha(j) = i$. Suppose the toroidal vector joining $A_{\alpha(i),i}$ to $A_{\alpha(k),k}$ is repeated as the toroidal vector joining $A_{\alpha(j),j}$ to $A_{\alpha(l),l}$, with i, j, k, l all distinct. Then we can rearrange the equation $((k - i) \bmod n, (\alpha(k) - \alpha(i)) \bmod n) = ((l - j) \bmod n, (\alpha(l) - \alpha(j)) \bmod n)$ to get $((j - i) \bmod n, (\alpha(j) - \alpha(i)) \bmod n) = ((l - k) \bmod n, (\alpha(l) - \alpha(k)) \bmod n)$. We say that another pair of repeated toroidal vectors is *induced*. Visually, we connect the initial endpoints and terminal endpoints of a pair of repeated toroidal vectors to create a parallelogram as shown in Figure 3 (provided the initial and terminal endpoints are all distinct).

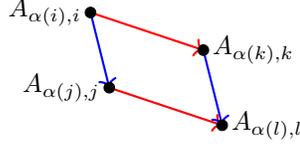


Figure 3: Induced repeats

We now formalize this idea using a graphical representation. The *graph of repeats* $G = (V, E)$ corresponding to permutation array A has vertex set $V = [n]$ where vertex i corresponds to the non-zero entry $A_{\alpha(i),i}$ in column i . Edge ij is in E if and only if the toroidal vector from $A_{\alpha(i),i}$ to $A_{\alpha(j),j}$ is repeated in A . Note G is undirected since toroidal vector (w, h) from $A_{\alpha(i),i}$ to $A_{\alpha(j),j}$ is repeated in A if and only if toroidal vector $(n-w, n-h)$ from $A_{\alpha(j),j}$ to $A_{\alpha(i),i}$ is repeated in A . In this way Figure 3 simplifies to:

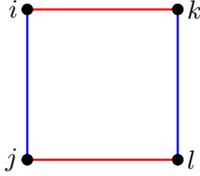


Figure 4: Graphical representation of Figure 3

5.1 Repeats Induced by Repeated Toroidal Vector $(\frac{n}{2}, \frac{n}{2})$

Let T be the T-matrix of A . We first consider the repeats induced by the repeated toroidal vector $(\frac{n}{2}, \frac{n}{2})$, because in this case the toroidal vectors (w, h) and $(n-w, n-h)$ are identical. By property (iii) of the T-matrix we may write $T_{\frac{n}{2}, \frac{n}{2}} = 2m$. The cases $m = 1, 2, 3$ are represented in Table 4. Arithmetic in the table is mod n . Edges are of the same colour if the toroidal vectors they correspond to are the same. The red edges correspond to toroidal vectors $(\frac{n}{2}, \frac{n}{2})$. We assume that the induced repeated toroidal vectors do not occur elsewhere in A , so that the non-red entries in the T-matrix corresponding to these repeats are exactly 2. (If a pair of toroidal vectors (w', h') is induced, and another (w', h') toroidal vector occurs elsewhere in A , then the (w', h') entry of T would exceed 2. Such an example is given in Section 5.5.)

In the case $m = 2$, two pairs of repeated toroidal vectors are induced. A first (blue) pair arises as described in Figure 4 (taking $k = i + \frac{n}{2}$ and $l = j + \frac{n}{2}$). However, since the toroidal vector $(\frac{n}{2}, \frac{n}{2})$ from $A_{\alpha(i),i}$ to $A_{\alpha(i+\frac{n}{2}),i+\frac{n}{2}}$ is the same as the toroidal vector $(\frac{n}{2}, \frac{n}{2})$ from $A_{\alpha(i+\frac{n}{2}),i+\frac{n}{2}}$ to $A_{\alpha(i),i}$, we may reverse this toroidal vector to induce a second (green) pair of repeated toroidal vectors. For general m , the graph of repeats contains a K_{2m} subgraph. Any two distinct vertices in the subgraph are terminal endpoints of two distinct $(\frac{n}{2}, \frac{n}{2})$ toroidal vectors, and so an edge is drawn between every pair of vertices.

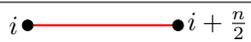
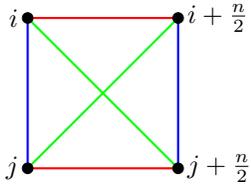
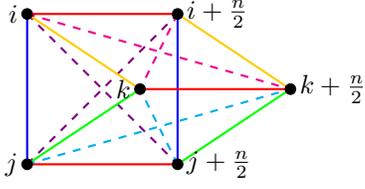
Case	Graph of Repeats	Repeats in T-matrix
$m = 1$		2
$m = 2$		4, 2, 2, 2, 2
$m = 3$		6, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2

Table 4: Cases of induced repeats when $T_{\frac{n}{2}, \frac{n}{2}} = 2m$

5.2 Repeats Induced by a 2 in the T-matrix

We now consider repeats induced by toroidal vectors other than $(\frac{n}{2}, \frac{n}{2})$, and in view of the analysis above we may assume that none of the induced repeated toroidal vectors is $(\frac{n}{2}, \frac{n}{2})$.

Suppose $T_{w,h} = 2$. Then the number of repeated toroidal vectors induced by the two (w, h) toroidal vectors depends on whether or not their corresponding edges are adjacent in the graph of repeats, as illustrated in Table 5. The entries for number of repeats in the T-matrix again assume that the induced repeated toroidal vectors do not occur elsewhere in A .

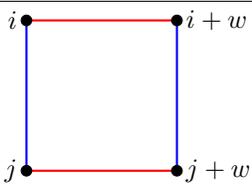
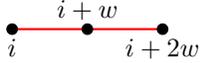
Case	Graph of Repeats	Repeats in T-matrix
1: Red edges are not adjacent.		2, 2, 2, 2
2: Red edges are adjacent.		2, 2

Table 5: Cases of induced repeats when $T_{w,h} = 2$

Case 1 is exactly as described in Figure 4. We say that these edges form a *parallelogram* in the graph of repeats. In case 2, since the edges corresponding to the two (w, h) toroidal vectors are adjacent, no repeats are induced.

5.3 Repeats Induced by a 3 in the T-matrix

If a toroidal vector appears more than two times in A , then a greater number of repeated toroidal vectors can be induced. For instance, consider Table 6 which illustrates cases of possible induced

repeats when a toroidal vector $(w, h) \neq (\frac{n}{2}, \frac{n}{2})$ occurs three times. There are $\binom{3}{2}$ possible pairs of repeats that can be induced (since we can possibly form $\binom{3}{2}$ parallelograms in the graph). We again assume that the induced repeated toroidal vectors are not $(\frac{n}{2}, \frac{n}{2})$, and that the induced repeated toroidal vectors do not occur elsewhere in A .

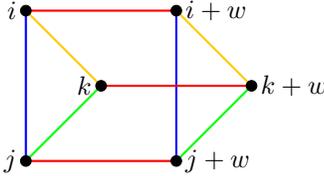
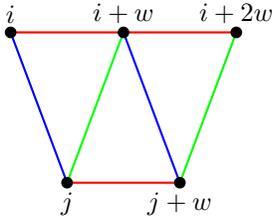
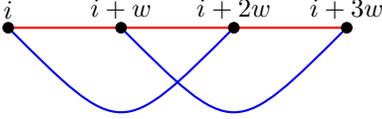
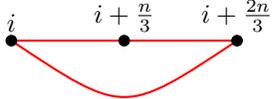
Case	Graph of Repeats	Repeats in T-matrix
1: No red edges are adjacent.		3, 3, 2, 2, 2, 2, 2, 2
2: One pair of red edges adjacent.		3, 3, 2, 2, 2, 2
3: Two pairs of red edges adjacent.		3, 3, 2, 2
4: Each pair of red edges adjacent.		3, 3

Table 6: Cases of induced repeats when $T_{w,h} = 3$

In case 1, all three pairs of red edges are not adjacent, and so we have three parallelograms in the graph; in case 2, two pairs of red edges are not adjacent, giving rise to two parallelograms; in case 3, only one parallelogram is induced by the one pair of red edges that are not adjacent. Note that in case 4, the red toroidal vector (w, h) must equal $(\frac{n}{3}, \frac{n}{3})$.

5.4 The Graph of Repeats of An Extended Golomb Costas Array

Consider the following example of a permutation array of order 10 given in Figure 5 corresponding to permutation $[1, 2, 4, 9, 8, 5, 3, 7, 10, 6]$. Its deficiency achieves the lower bound $10 - 3 = 7$. Also consider its T-matrix T and its graph of repeats $G = (V, E)$ in Figures 6 and 7 respectively. Two edges are coloured the same in G if their corresponding toroidal vectors are the same or are induced by each other (i.e. parallelograms in G are coloured the same). Also, if an edge $e \in E$ corresponds to toroidal vector (w, h) , then the (w, h) entry of T has the same colour as e . For example, the two $(2, 7)$ toroidal vectors induce the two $(5, 1)$ toroidal vectors as shown in Figure 5, and the entries in T and edges in G corresponding to these toroidal vectors are both coloured blue. This colouring uncovers relationships between repeats in the T-matrix.

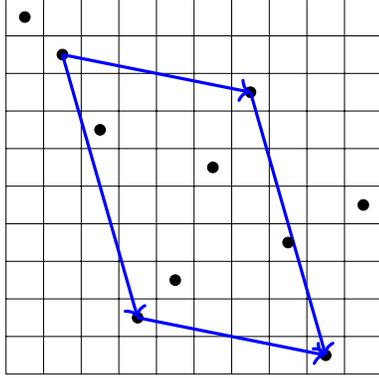


Figure 5: Permutation array A

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \mathbf{2} & \mathbf{2} & 0 & 1 & 1 \\ 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 0 & \mathbf{2} & 1 \\ 1 & 1 & \mathbf{2} & 1 & 1 & 0 & 1 & 1 & \mathbf{2} & 1 \\ \mathbf{2} & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \mathbf{2} \\ 1 & \mathbf{2} & 1 & 0 & 1 & 1 & 1 & \mathbf{2} & 1 & 1 \\ 1 & \mathbf{2} & 0 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 1 \\ 1 & 0 & \mathbf{2} & \mathbf{2} & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \mathbf{2} & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 6: T-matrix of A

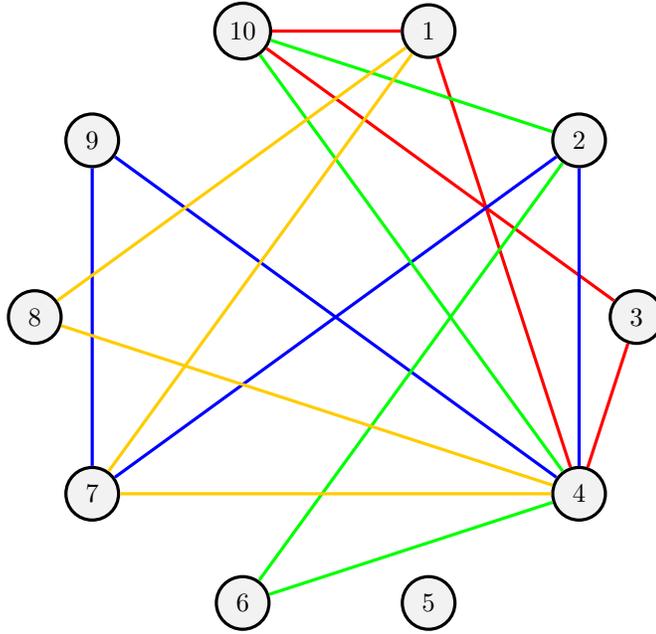


Figure 7: Graph of repeats of A

Observe that no two edges corresponding to the same repeated toroidal vector are adjacent, following case 1 of Table 5 (i.e. every edge is part of a parallelogram), and that vertex 4 is part of every parallelogram. If vertex 4 is removed from G , then $E = \emptyset$.

Recall from Section 2 that all entries of the T-matrix of an augmented Golomb Costas array G_+^* are either 0 or 1. Thus all entries greater than 1 in the T-matrix of an extended Golomb Costas array G' , or equivalently, all repeats in G' , are introduced by the additional non-zero entry of G' . In the example above, the non-zero entry of A corresponding to vertex 4 is the additional non-zero entry of an extended Golomb Costas array responsible for all repeated toroidal vectors. All tested examples of permutation arrays of order $n \in \{4, 6, 10, 12, 16, 18\}$ with deficiency $n - 3$ and all tested examples of symmetric permutation arrays of order $n \in \{22, 24, 28, 30\}$ with deficiency $n - 3$ have a graph of repeats with a similar structure as the example graph above.

5.5 Applying the Graph of Repeats to the Gap Structure

Next we have another example of a permutation array of order 10 but with deficiency exceeding the lower bound. Permutation array A corresponds to permutation $[1, 2, 4, 5, 3, 7, 10, 9, 6, 8]$ and has deficiency 14 (the third smallest attainable deficiency over permutation arrays of order 10, as seen by the gap structure in Table 3). Its T-matrix and graph of repeats are in Figures 9 and 10 respectively. Edges are coloured the same if the toroidal vectors they represent are the same or are somehow induced by each other. Even though there are many repeated toroidal vectors in A , only four colours are needed because the toroidal vectors giving rise to the 4 and pair of 3s in the T-matrix induce many other repeats.

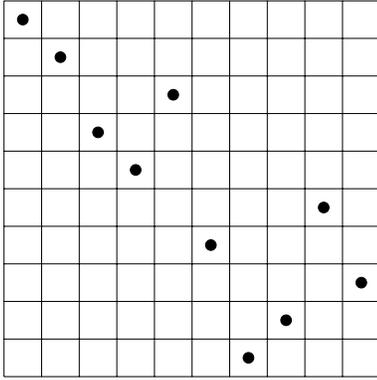


Figure 8: Permutation array A

$$\begin{pmatrix} 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 4 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}$$

Figure 9: T-matrix of A

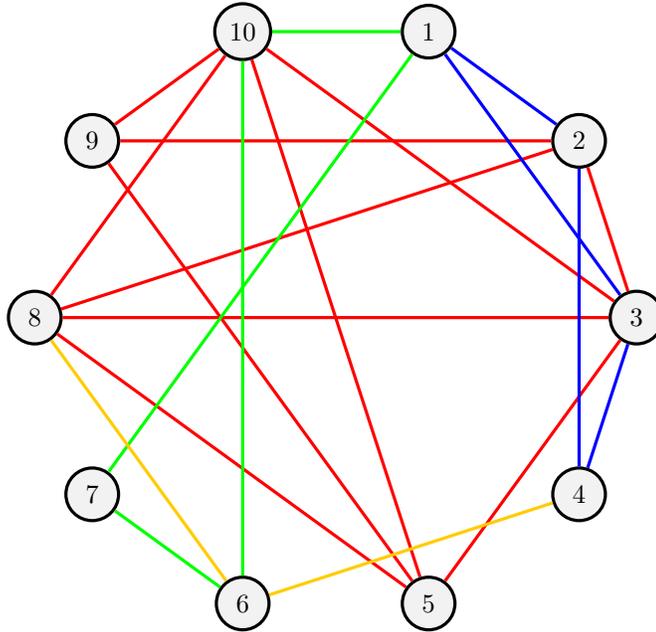


Figure 10: Graph of repeats of A

The green and blue edges follow case 1 in Table 5, and the yellow edges follow case 2 in Table 5.

Let’s take a closer look at the red edges in the graph (the repeats induced by or related to the four $(5, 5)$ toroidal vectors), which have been recoloured in Figure 11.

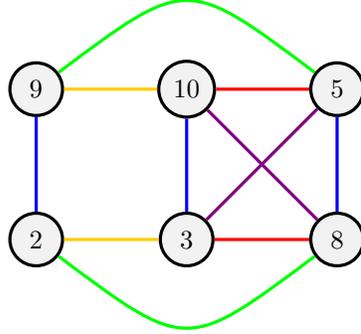


Figure 11: A closer look at the red repeats

The red edges in Figure 11 correspond to the two pairs of $(5, 5)$ toroidal vectors involving vertices 3, 5, 8, and 10. These toroidal vectors induce a pair of purple $(2, 9)$ toroidal vectors and a pair of blue $(3, 6)$ toroidal vectors, resulting in a K_4 subgraph on vertices 3, 5, 8, and 10. But there is another blue $(3, 6)$ toroidal vector between the non-zero entries in columns 2 and 9 in A , giving rise to a third blue edge in Figure 11. Each pair of blue edges are not adjacent (case 1 in Table 6), and so we have $\binom{3}{2}$ parallelograms formed among the three blue edges. Thus the two yellow $(1, 2)$ toroidal vectors and the two green $(4, 3)$ toroidal vectors are induced as well.

6 Conclusion

The graph of repeats highlights how repeated toroidal vectors can occur in large clusters. Perhaps, when the lower bound deficiency $n - 3$ is achievable, and when trying to take a small step up from minimum deficiency, this accumulation of repeats forces the step to be large, resulting in the gap structure we explain in Section 4.

Our discovery of the gap structure and our exploration of how repeated toroidal vectors are induced through the graph of repeats data structure led us to amend our conjecture to Conjecture 1.11, namely:

$$D_n = n - 3 \text{ if and only if } n = p^r - 1, \text{ where } p > 3 \text{ is prime.}$$

References

- [1] S. Golomb, “Algebraic constructions for Costas arrays,” *J. Combinat. Theory, Ser. A*, vol. 37, no. 1, pp. 13–21, 1984.
- [2] J. Jedwab and J. Wodlinger, “The deficiency of Costas arrays,” *IEEE Trans. Inf. Theory*, vol. 60, no. 12, pp. 7947–7954, Dec. 2014.
- [3] D. Panario, A. Sakzad, B. Stevens, and Q. Wang, “Two new measures for permutations: Ambiguity and deficiency,” *IEEE Trans. Inf. Theory*, vol. 57, no. 11, pp. 7648–7657, Nov. 2011.
- [4] D. Panario, B. Stevens, and Q. Wang, “Ambiguity and deficiency in Costas arrays and APN permutations,” *LATIN: Theoretical Informatics (Lecture Notes in Computer Science)*, vol. 6034, A. López-Ortiz, Ed. Berlin, Germany: Springer-Verlag, 2010, pp. 397–406.

Minimum Deficiency of Permutation Arrays

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Questions and Initial Conjecture

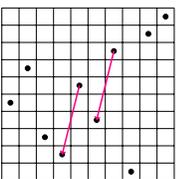
Q: For which even n can the **minimum deficiency** of a permutation array of order n achieve the lower bound $n - 3$?

Conjecture: Exactly when $n \equiv 0$ or $4 \pmod{6}$.

Q: What are helpful **data structures** for studying the deficiency of permutation arrays?

Key Concepts

Let A be a permutation array of order n . The (i, j) entry of its **T-matrix** T is given by the # of times the periodic vector (i, j) appears in A .
The **deficiency** of A is the # of zeros in T .



Permutation array A

T-matrix of A

1	1	1	1	2	1	1	1	1	1
1	2	1	1	1	0	1	1	2	1
1	1	1	1	1	2	1	2	0	1
2	0	1	1	1	2	1	1	1	1
1	1	2	1	0	1	2	1	1	1
1	1	1	2	1	1	1	0	1	1
0	2	1	2	1	1	1	1	1	1
2	1	1	0	1	1	1	1	2	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	2	1	1	1	1	1

Let D_n be the minimum deficiency over all permutation arrays of even order n . The example above shows $D_{10} \leq 7$.

References

- [1] D. Panario, A. Sakzad, B. Stevens, and Q. Wang. "Two new measures for permutations: Ambiguity and deficiency." *IEEE Trans. Inf. Theory*, 57(11):7648–7657, 2011.
 - [2] D. Panario, B. Stevens, and Q. Wang. "Ambiguity and deficiency in Costas arrays and APN permutations." *LATW 2010: Theoretical Informatics (A. López-Ortiz, ed.) Lecture Notes in Computer Science* vol. 6034, Springer, Berlin, 2010, pp. 397–406.
 - [3] J. Jedwab and J. Woodlinger. "The deficiency of Costas arrays." *IEEE Trans. Inf. Theory*, 60(12):7947–7954, 2014.
- This project is funded by NSERC. Many thanks to my supervisor Dr. Jonathan Jedwab.

Motivating Observations

There is a known lower bound on D_n .

Theorem ([1], [2]).
 $D_n \geq n - 3$, with equality if $n = p^r - 1$ for $p > 3$ prime.

Can the lower bound be achieved?

YES NO MAYBE

n	D_n	n	D_n
4 ($= 5 - 1$)	1	20	?
6 ($= 7 - 1$)	3	22 ($= 23 - 1$)	19
8 ($= 3^2 - 1$)	6	24 ($= 5^2 - 1$)	21
10 ($= 11 - 1$)	7	26 ($= 3^3 - 1$)	23 or 24
12 ($= 13 - 1$)	9	28 ($= 29 - 1$)	25
14	25	30 ($= 31 - 1$)	27
16 ($= 17 - 1$)	13	32	?
18 ($= 19 - 1$)	15	34	?

Exhaustive search Symmetric search

- For all tested red permutation arrays, we noticed patterns in the T-matrix:
 - ↪ Cycles of 2s involving 3 columns and 3 rows
 - ↪ "Evenly" spaced zeros
 - ↪ No entry ≥ 3 if $n \equiv 4 \pmod{6}$
 - ↪ Exactly two 3s and no entry > 3 if $n \equiv 0 \pmod{6}$.
- For all red n , there is a minimum deficiency permutation array that is symmetric.

BUT it turns out...

- ↪ ALL tested red permutation arrays arise when $n = p^r - 1$, $p > 3$ prime, by modifying a Costas array construction ([1],[3]).
- ↪ ALL the apparent patterns in the associated T-matrix can be derived as a consequence!
- ↪ So should we amend the conjecture to "Exactly when $n = p^r - 1$, $p > 3$ prime?"
- ↪ **Smallest test case: Is $D_{34} = 31$?**
- ↪ Program running to search symmetric order 34 permutations; estimate 3 months to finish.

Search Algorithm

Goal: Find all permutation arrays A of order n having deficiency no greater than a target d .

Strategy:

- ↪ Set $A_{1,1} = 1$ (cyclic rotation of rows and columns preserves deficiency).
- ↪ Recursively assign each entry in A and determine the periodic vectors introduced by this new entry.
- ↪ Maintain a count of the number of repeated periodic vectors in A , including multiplicity.
- ↪ Terminate branch of recursion if the number of repeats forces the deficiency of A to exceed d .

Findings: Gap Structure

n	Attainable deficiency
4	1, 6
6	3, 4, ...
8	6, 9, 10, ...
10	7, 12, 14, 15, ...
12	9, 17, 19, 20, ...
14	25, 26, ...
16	13, 31, 33, 34, ...

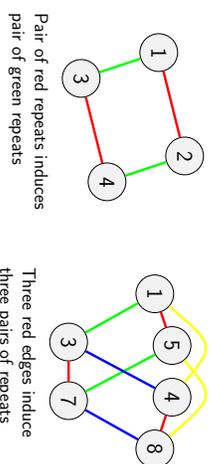
Exhaustive search results

Q: What breaks when trying to take a small step up from the lower bound?

Q: Do these gaps offer clues as to when $D_n = n - 3$?

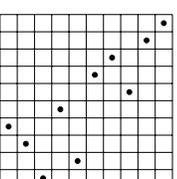
New Data Structure: Graph of Repeats

Let $A = (A_{i,j})$ correspond to permutation $\alpha(j) = i$. Draw an edge between vertices i and j in the graph of A if the periodic vectors between $A_{\alpha(i)j}$ and $A_{\alpha(j)i}$ are repeated elsewhere in A . This structure allows us to explore the **induction of repeats**.



How the T-matrix & Graph Interact

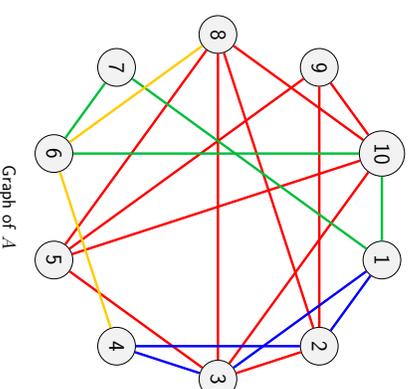
Consider the following example of a permutation of order 10, deficiency 14, and its T-matrix and graph. Edges are coloured according to induction of repeats.



Permutation array A

T-matrix of A

2	2	2	1	0	0	1	1	1	1
0	2	2	1	1	1	0	1	2	1
1	1	1	1	3	0	1	1	1	1
2	1	2	1	1	1	1	1	0	1
1	1	0	1	4	1	0	1	1	1
0	1	1	1	1	2	1	1	2	1
1	1	0	3	1	1	1	1	1	1
2	0	1	1	1	1	2	2	0	1
1	1	1	0	1	2	2	2	2	2



The graph of repeats gives a visualization of relationships between entries of the T-matrix.

Q: How can we constrain the deficiency of A via its graph of repeats?

Amended Conjecture

Having realized our initial observations were misleading, we sought a new approach. Exploration of the gap structure and graph of repeats led to:

Amended Conjecture.
 $D_n = n - 3$ if and only if $n = p^r - 1$ for $p > 3$ prime.