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Blow-up in a fourth-order semilinear parabolic equation from explosion-convection theory

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We describe various blow-up patterns for the fourth-order one-dimensional semilinear parabolic equation

$$u_t = -u_{xxxx} + \beta[(u_x)^3]_x + e^u$$

with a parameter $\beta \geqslant 0$, which is a model equation from explosion-convection theory. Unlike the classical Frank-Kamenetskii equation $u_t = u_{xx} + e^u$ (a solid fuel model), by using analytical and numerical evidence, we show that the generic blow-up in this fourth-order problem is described by a similarity solution $u_*(x,t) = -\ln(T-t) + f_1(x/(T-t)^{1/4})$ ($T > 0$ is the blow-up time), with a non-trivial profile $f_1 \not\equiv 0$. Numerical solution of the PDE shows convergence to the self-similar solution with the profile f_1 from a wide variety of initial data. We also construct a countable subset of other, not self-similar, blow-up patterns by using a spectral analysis of an associated linearized operator and matching with similarity solutions of a first-order Hamilton–Jacobi equation.

1 Introduction

The role of self-similar solutions is well known in second-order reaction-diffusion equations that exhibit finite time blow-up. Because of their importance in physical applications, many canonical equations such as the *semilinear* one-dimensional Frank-Kamenetskii equation [11] (the solid fuel model [35])

$$u_t = u_{xx} + e^u \tag{1.1}$$

and its counterpart with a power nonlinearity

$$u_t = u_{xx} + u^p, \quad p > 1, \tag{1.2}$$

have been well studied for the past thirty years. The single point formation of blow-up singularities was known from the 1970s [21], and was rigorously established in the 1990s [2, 10, 20, 27, 32, 33]; see also the references in a survey by Galaktionov & Vazquez [17]. While exact self-similar solutions are known to exist for second-order reaction-diffusion *quasilinear* problems [3, 28],

$$u_t = (|u_x|^\sigma u_x)_x + e^u \quad \text{or} \quad u_t = (u^\sigma u_x)_x + u^p \quad \text{with} \ \ \sigma > 0,$$

it has been shown that none exist for the above semilinear problems (1.1), (1.2), where the asymptotic blow-up behaviour is described by approximate similarity solutions satisfying first-order Hamilton–Jacobi equations. Therefore, it is important to discover if semilinear fourth-order parabolic equations exhibit similar behaviour to their second-order counterparts and not possess exact self-similar solutions due to the *semilinear* structure of both problems.

In this paper, we show that this is not the case for a model from explosion-convection theory [23]

$$u_t = -u_{xxxx} - [(2 - (u_x)^2)u_x]_x - \alpha u + qe^{su}, \tag{1.3}$$

where α, q and s are certain positive constants obtained from physical parameters. More precisely, we construct various self-similar and approximate self-similar blow-up patterns for the following large solution reduction of this equation:

$$u_t = -u_{xxxx} + \beta[(u_x)^3]_x + e^u \tag{1.4}$$

of which the fourth-order extension of the Frank-Kamenetskii equation,

$$u_t = -u_{xxxx} + e^u, \tag{1.5}$$

is a limiting case.

Our main goal is to present a description of a countable family of different blow-up patterns occurring in equation (1.4). The first two of these are exact self-similar solutions (nonlinear patterns), and the rest are composed of linearized patterns constructed via spectral analysis, invariant manifold theory and matched asymptotic expansions. For each such linearized pattern, which can actually occur for special initial data, we specify the asymptotic evolution and calculate the spectrum of admissible final-time profiles.

Thus, unlike the second-order classical combustion models (1.1) and (1.2), where nonlinear blow-up similarity patterns do not exist at all, in higher-order blow-up models there are nonlinear patterns and these patterns are expected to describe the evolution of generic initial data. The remainder of the family of countable patterns is constructed via linearization, spectral analysis and matching as are all the patterns in the classical models. We expect that this is a general and essentially new feature of $2m$th order ($m > 1$) reaction-diffusion-convection semilinear parabolic models.

While the discrete spectrum of linearized blow-up patterns is constructed by a known invariant manifold theory, the nonlinear similarity patterns are studied by numerical methods and some formal analytic estimates only. We show that blow-up analysis of higher-order semilinear equations leads to a number of new hard ODE and PDE problems associated with non self-adjoint and non-potential operators. We state and conjecture the results on key related open problems.

In the rest of this section, we describe the derivation, validity and some auxiliary properties of our model equation and introduce the relevant mathematical definitions and rescaled equations. In particular, we show in what sense our reduced model (1.4) is reasonable. In § 2 we present necessary properties of the linearized differential operator which governs the dynamics of the rescaled equations.

In § 3 we present formal analytical and numerical evidence which indicates that there are two non-trivial self-similar solutions to the reduced equation (1.4) for different values of the parameter $\beta \geqslant 0$. Moreover, numerical simulations of both equations show that the 'ground state', which is monotone decreasing in $|x|$, is asymptotically stable in the rescaled sense. The solution to the reduced equation captures the blow-up dynamics of the full model equation.

These results are in striking contrast with properties of blow-up for the second-order semilinear equations, where the generic singularity behaviour is only approximately self-similar, and is instead described by similarity solutions of the first-order Hamilton–Jacobi equations and is obtained by a matched asymptotic analysis (see the recent survey by Galaktionov & Vazquez [17]).

In § 4, by a standard invariant manifold theory, we construct the rest of the countable subset of different blow-up patterns by using matched asymptotic expansions and spectral properties of non self-adjoint ordinary differential operators obtained in § 2.

1.1 The Semenov–Rayleigh–Benard problem

The fourth-order one-dimensional semilinear parabolic equation (1.3) occurs in the Semenov–Rayleigh–Benard problem [23], where the equation is derived in studying the interaction between natural convection and the explosion of an exothermically-reacting fluid confined between two isothermal horizontal plates. This is an evolution equation for the temperature fluctuations in the presence of natural convection (it is assumed that the Rayleigh number is marginally supercrtical), small wall losses and chemistry. It can be considered as a formal combination of the equation derived in Gertsberg & Shivashinsky [18] (see also Chapman & Proctor [5], where the derivation is similar except for the absence of reactive effects) for the Rayleigh–Benard problem and of the Semenov-like energy balance [12, 29], showing that natural convection and the explosion mechanism may reinforce each other when the convection cells are large enough. In this statement, solutions $u = u(x,t)$ are 2π-periodic in x with bounded periodic initial data $u(x,0) = u_0(x)$ for $x \in I_0 = [0, 2\pi]$. The previous authors [23] were concerned with the stability of small solutions and the possibility of thermal runaway, and they did not consider the spatio-temporal structure of blow-up solutions. Equation (1.3) can be treated as a higher-order generalization of the Frank-Kamenetskii equation (1.1).

Equation (1.3) is uniformly parabolic with all spatial differential operators appearing in divergence form. It admits a unique classical, local in time, solution (see the standard parabolic theory in Eidelman [9], Friedman [13] and Taylor [31]). The operator on the right-hand side of (1.3)

$$\mathbf{N}(u) \equiv -u_{xxxx} - [(2-(u_x)^2)u_x]_x - \alpha u + qe^{su}$$

is potential, and the equation admits the Lyapunov function

$$L[u](t) = \int_{I_0} \left[\frac{1}{2}(u_{xx})^2 - (u_x)^2 + \frac{1}{4}(u_x)^4 + \frac{\alpha}{2}u^2 - \frac{q}{s}e^{su} \right] dx,$$

which is monotone on bounded orbits,

$$\frac{d}{dt}L[u](t) = -\int_{I_0} (u_t)^2 dx \leqslant 0.$$

For such smooth gradient systems, the ω-limit set of any bounded orbit $\{u(\cdot,t),\, t > 0\}$,

$$\omega(u_0) = \{f \in C(I_0) : \; \exists \{t_k\} \to \infty \text{ such that } u(\cdot,t_k) \to f \text{ uniformly}\},$$

is known to consist of stationary solutions: $\mathbf{N}(f) = 0$ in I_0 for any $f \in \omega(u_0)$. Some results on bifurcation of stationary solutions were obtained in Joulin *et al.* [23]. If the subset of stationary solutions consists of isolated equilibria, the asymptotic behaviour of uniformly bounded orbits does not essentially differ from that for the second-order parabolic equation (1.1) where any bounded orbits are known to approach a stationary profile as $t \to \infty$.

1.2 Finite time blow-up solutions and similarity variables

One of the central concepts to reaction-diffusion equations is finite time blow-up (meaning explosion in combustion theory [35]), where the solution becomes unbounded at some time, $T > 0$, in the sense that $u(x,t)$ exists and is classical on any time-interval $[0, T']$ with $T' \in (0, T)$ and

$$\sup_x |u(x,t)| \to \infty \quad \text{as } t \to T^-. \tag{1.6}$$

The fact that solutions of higher-order parabolic equations may blow-up is well known [6, 8, 17, 25] for equations $u_t = -(-\Delta)^m u + f(u)$, where the critical Fujita exponent and estimates on blow-up rates were established for any $m > 1$. The proof of blow-up for the periodic initial value problem for (1.3) is straightforward.

Proposition 1 *Let* $u(x,t)$ *be a solution of* (1.3) *in* $I_0 \times \mathbf{R}_+$ *with periodic boundary conditions and bounded continuous initial data* $u_0(x)$. (i) *If*

$$\alpha < sqe^{1/2\pi}, \tag{1.7}$$

then the solution blows up in finite time. (ii) *If* (1.7) *does not hold, then the solution blows up if initial data is sufficiently large in the mean sense.*

Proof Denoting by $\bar{u}(t) = \int u(x,t)dx$ the mean of the solution on I_0 and integrating equation (1.3) over I_0, we obtain

$$\bar{u}' = q\int e^{su}\, dx - \alpha\bar{u}.$$

By Jensen's inequality for convex functions

$$\int e^{su}\, dx = 2\pi \int e^{su}\frac{1}{2\pi}\, dx \geqslant 2\pi e^{s\bar{u}/2\pi},$$

and we arrive at an ordinary differential inequality

$$\bar{u}' \geqslant F(\bar{u}) \equiv 2\pi q e^{s\bar{u}/2\pi} - \alpha\bar{u}. \tag{1.8}$$

If (1.7) holds, then $F > 0$ in $\mathbf{R}$. Hence, $\bar{u}' > 0$, $\bar{u}(t) > \bar{u}_0$ for $t > 0$ and (1.8) implies that $\bar{u}(t)$ (and $u(x,t)$) blows up at

$$T \leqslant T_0 = \int_{\bar{u}_0}^{\infty} ds/F(s).$$

(ii) If $\alpha \geqslant sqe^{1/2\pi}$, then similarly we have that blow-up occurs if $\bar{u}_0 > s_+$, where s_+ is the maximal root of the equation $F(s) = 0$. □

Blow-up is an essential feature of explosion-convection problems and the corresponding parabolic equations under consideration. As for the solid fuel model (1.1), the structure of such a blow-up singularity formation is of importance in the present higher-order model. Finite time blow-up involves a delicate balance between the spatial and temporal derivatives and the reaction terms driving the blow-up. This balance is made naturally apparent by considering the scaling invariance of the underlying PDE. This scaling structure is also important for the numerical methods employed in integrating the full PDE (see § 3).

For convenience, we rescale $su \mapsto u$, $(qs)^{1/4}x \mapsto x$ and $qst \mapsto t$ to obtain the equation

$$u_t = \mathbf{A}(u) - \gamma u_{xx} - \delta u, \quad \gamma = 2/\sqrt{qs}, \;\; \delta = \alpha/qs, \tag{1.9}$$

where $\mathbf{A}$ is the operator

$$\mathbf{A}(u) = -u_{xxxx} + \beta[(u_x)^3]_x + e^u, \quad \beta = 1/s^2.$$

Here β is an essential parameter which cannot be removed by scaling. The physically admissible range of the parameter is $\beta \in (0, \infty)$, but we also include the limit case $\beta = 0$ which formally corresponds to $s = \infty$ and leads to the fourth-order extended Frank-Kamenetskii equation (1.5).

We begin our similarity analysis with the unperturbed equation (1.4) which is a natural simplification of (1.9) for large solutions as only the lowest order terms have been neglected.

Without loss of generality, we assume that the solution $u(x,t)$ blows up at finite time $t = T$ in the sense of (1.6) and the blow-up set

$$B[u_0] = \{x \in I: \text{ there exist } \{x_k\} \to x, \; \{t_k\} \to T^- \text{ such that } u(x_k, t_k) \to \infty\} \tag{1.10}$$

contains the origin, $0 \in B[u_0]$. We now observe that (1.4) is invariant under the group of transformations

$$t \mapsto \lambda t, \quad x \mapsto \lambda^{1/4}x, \quad u \mapsto u - \ln\lambda, \quad \text{with } \lambda > 0.$$

Therefore, motivated by the blow-up assumptions, replacing $t \mapsto t - T$ and setting $\lambda = (T-t)^{-1}$ yield the following independent self-similar variables: $y = x/(T-t)^{1/4} : I_0 \to \mathbf{R}$

is the new spatial variable, and $\tau = -\ln(T-t) : (0, T) \to (\tau_0, \infty)$ with $\tau_0 = -\ln T$ is the new time variable. Then the rescaled solution is given by

$$u(x,t) = -\tau + \theta(y,\tau), \tag{1.11}$$

and substituting into (1.4) gives the rescaled equation

$$\theta_\tau = -\theta_{yyyy} + \beta[(\theta_y)^3]_y - \frac{1}{4} y\theta_y + e^\theta - 1 \equiv \mathbf{A}_1(\theta). \tag{1.12}$$

Using these new variables for the full equation (1.3) gives that the rescaled function θ satisfies the following perturbed parabolic equation:

$$\theta_\tau = \mathbf{A}_1(\theta) + \mathbf{C}(\theta,\tau), \tag{1.13}$$

where $\mathbf{A}_1$ is the autonomous operator above and $\mathbf{C}$ is a non-autonomous perturbation,

$$\mathbf{C}(\theta,\tau) = -\gamma e^{-\tau/2}\theta_{yy} - \delta e^{-\tau}(\tau + \theta),$$

which is exponentially small as $\tau \to \infty$ (i.e. as $t \to T^-$) on bounded orbits.

The Cauchy problem for single point blow-up. It follows from the scaling variable y in (1.11) that for arbitrarily small fixed $|x| > 0$, the corresponding $|y| \to \infty$ as $\tau \to \infty$ which, in general, corresponds to formation of a *single point singularity* (see further comments below). This means a strong localization phenomenon and that the type of boundary conditions in the periodic, Dirichlet or Neumann boundary value problem for (1.3) is irrelevant.

Therefore, it is natural to consider the *Cauchy problem* for equation (1.13) with bounded initial data at $\tau = \tau_0$

$$\theta_0(y) = u_0\big(T^{1/4}y\big) - \tau_0 \quad \text{in } \mathbf{R}.$$

The perturbed equation (1.13) suggests that we consider first the unperturbed rescaled equation

$$g_\tau = \mathbf{A}_1(g) \quad \text{in } Q_0 = \mathbf{R} \times (\tau_0, \infty), \quad g(y,\tau_0) = g_0(y) \quad \text{in } \mathbf{R}. \tag{1.14}$$

According to (1.11), $g(y,\tau)$ is simply the rescaled solution of (1.4).

Any solution to (1.4) may be rescaled to become a solution to (1.12). However, only exactly (rather than asymptotically) self-similar solutions have θ independent of τ. Hence, we begin with construction of blow-up self-similar solutions of (1.4) of the form

$$u_*(x,t) = -\ln(T-t) + f(y), \quad y = x/(T-t)^{1/4}, \tag{1.15}$$

where f satisfies the following ODE:

$$\mathbf{A}_1(f) = 0 \quad \text{in } \mathbf{R}_+ \tag{1.16}$$

supplemented with the symmetry conditions

$$f'(0) = 0, \quad f'''(0) = 0$$

at the origin. The symmetry assumptions are quite natural in blow-up analysis for various second-order parabolic equations, and as is typical for blow-up problems, for stable (generic) blow-up profiles, are connected with the idea of infinite time symmetrization (as $\tau \to \infty$) in parabolic equations like (1.12) and (1.13). We expect that these should be kept for higher-order equations, but we cannot prove that the ODE (1.16) does not admit suitable non-symmetric profiles. We note that for second-order equations, the proof of eventual symmetrization is usually done using Alexandrov's Reflection Principle, moving plane techniques and other approaches based on the Maximum Principle, which are not available for any $m > 1$. Due to the exponential nonlinearity, purely anti-symmetric solutions are not permitted by the ODE.

The ODE (1.16) has a two-parametric bundle of admissible profiles $f(y)$ at infinity,

$$f(y) = [-4\ln|y| + C + o(1)] + C_1 y^{-1/3} e^{-a_0 y^{4/3}} [1 + o(1)] \quad \text{as } y \to \infty, \tag{1.17}$$

where $a_0 = 3/4^{4/3}$ and $C, C_1 \in \mathbf{R}$ are parameters. The first one, C, determines the actual far field behaviour of the blow-up solution such that the limit profile $u_*(x, T^-)$ is bounded in a deleted neighbourhood of the origin $0 \in B[u_0]$. Specifically we have that for any $x > 0$, and any symmetric profile f, there exists a finite limit (i.e. the final-time profile)

$$\lim_{t\to T} u_*(x,t) = \lim_{t\to T} \left[-\ln(T-t) + f\left(\frac{x}{(T-t)^{1/4}} \right) \right] = -4\ln|x| + C. \tag{1.18}$$

The second parameter, C_1, has no role in the leading order structure of the solution, but specifies a complicated two-dimensional topology of the ODE solutions.

2 On spectral properties of the linearized operator

Consider the linearization of operator (1.12) about the zero solution

$$\mathbf{A}_1'(0) = \mathbf{B}^* + I, \quad \text{where } \mathbf{B}^* = -\frac{d^4}{dy^4} - \frac{1}{4} y \frac{d}{dy}, \tag{2.1}$$

where I denotes the identity operator and $'$ is the Frechet derivative. In this section we will describe necessary spectral properties of the linear operator $\mathbf{B}^*$ and its formal adjoint differential expression

$$\mathbf{B} = -\frac{d^4}{dy^4} + \frac{1}{4} y \frac{d}{dy} + \frac{1}{4} I. \tag{2.2}$$

Note that both operators are not symmetric, and do not admit a self-adjoint extension.

The spectral properties of these linear operators is crucial to the study of our families of blow-up patterns and the asymptotic analysis to follow so we now present some results from Egorov *et al.* [8] and Galaktionov [14].

2.1 Fundamental solution

We begin with the fundamental solution of the corresponding linear fourth-order parabolic operator. Consider the linear equation

$$u_t = -u_{xxxx}. \tag{2.3}$$

The fundamental solution has the standard self-similar form

$$b(x,t) = t^{-1/4} F(\eta), \quad \eta = x/t^{1/4}.$$

Substituting $b(x,t)$ into (2.3) and taking the Fourier transform yields that the even profile $F(\eta)$ is the unique even solution of the linear ODE

$$\mathbf{B}F = 0 \ \text{ in } \mathbf{R}.$$

Hence,

$$F(\eta) = \alpha \int_0^\infty e^{-s^4} \cos(s\eta)\, ds, \tag{2.4}$$

where α is chosen to normalize, $\int F = 1$,

$$\alpha = \left(\int_0^\infty \int_0^\infty e^{-s^4} \cos(s\eta)\, ds\, d\eta \right)^{-1}.$$

The rescaled kernel $F(\eta)$ satisfies a standard pointwise estimate [9]

$$|F(\eta)| \leqslant D e^{-d|\eta|^{4/3}} \quad \text{in } \mathbf{R},$$

where D and d are positive constants.

2.2 Discrete real spectrum and eigenfunctions of B

We study the spectrum $\sigma(\mathbf{B})$ in the weighted space $L^2_\rho(\mathbf{R})$ with the exponential weight

$$\rho(y) = e^{a|y|^{4/3}} > 0 \quad \text{in } \mathbf{R},$$

where $a \in (0, 2d)$ is a sufficiently small positive constant. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the corresponding inner product and the induced norm, respectively.

We next introduce a Hilbert space of functions $H^4_\rho(\mathbf{R})$ with the inner product

$$\langle v, w \rangle_\rho = \int_{\mathbf{R}} \rho(y) \sum_{k=0}^{4} D^k v(y)\, \overline{D^k w(y)}\, dy,$$

where $D = d/dy$, and the norm

$$\|v\|^2_\rho = \int_{\mathbf{R}} \rho(y) \sum_{k=0}^{4} |D^k v(y)|^2\, dy.$$

Then $H^4_\rho(\mathbf{R}) \subset L^2_\rho(\mathbf{R}) \subset L^2(\mathbf{R})$ and $\mathbf{B}$ is a bounded linear operator from $H^4_\rho(\mathbf{R})$ to $L^2_\rho(\mathbf{R})$. Let us present its spectral properties.

Lemma 1 *The spectrum of* $\mathbf{B}$ *consists of real simple eigenvalues only*

$$\sigma(\mathbf{B}) = \{\lambda_k = -k/4,\ k = 0, 1, 2, \ldots\}. \tag{2.5}$$

The eigenfunctions

$$\psi_k(y) = \frac{(-1)^k}{\sqrt{k!}} D^k F(y) \tag{2.6}$$

form a complete subset in $L^2(\mathbf{R})$ *and in* $L^2_\rho(\mathbf{R})$, *where* F *is as defined in* (2.4).

2.3 Polynomial eigenfunctions of the adjoint operator $\mathbf{B}^*$

We consider the formally adjoint operator (2.1) in the weighted space $L^2_{\rho^*}(\mathbf{R})$ ($(\cdot,\cdot)_*$ and $\|\cdot\|_*$ are the inner product and the norm) with the exponentially decaying weight function

$$\rho^*(y) \equiv \frac{1}{\rho(y)} = e^{-a|y|^{4/3}} > 0,$$

and ascribe to $\mathbf{B}^*$ the domain $H^4_{\rho^*}(\mathbf{R})$ dense in $L^2_{\rho^*}(\mathbf{R})$. Then

$$\mathbf{B}^* : H^4_{\rho^*}(\mathbf{R}) \to L^2_{\rho^*}(\mathbf{R})$$

is a bounded linear operator and $\mathbf{B}^*$ is adjoint to $\mathbf{B}$,

$$\langle \mathbf{B}v, w\rangle = \langle v, \mathbf{B}^* w\rangle \quad \text{for any } v \in H^4_\rho(\mathbf{R}),\ w \in H^4_{\rho^*}(\mathbf{R}).$$

Therefore, $\sigma(\mathbf{B}) = \sigma(\mathbf{B}^*)$ [24].

Lemma 2 *The eigenfunctions* $\psi^*_k(y)$ *of* $\mathbf{B}^*$ *are polynomials of order* k,

$$\psi^*_k(y) = \frac{1}{\sqrt{k!}} \sum_{j=0}^{\lfloor -\lambda_k \rfloor} \frac{1}{j!} D^{4j} y^k, \quad k = 0, 1, 2, \ldots, \tag{2.7}$$

and form a complete subset in $L^2_{\rho^*}(\mathbf{R})$. (Here $\lfloor\cdot\rfloor$ denotes the integer part.)

Integrating by parts, we have that the orthonormality condition holds

$$\langle \psi_k, \psi^*_l\rangle = \delta_{k,l} \quad \text{for any } k, l \geqslant 0, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ is the duality product in $L^2(\mathbf{R})$ and $\delta_{\beta,\gamma}$ is the Kronecker delta. Operators $\mathbf{B}$ and $\mathbf{B}^*$ have zero Morse index (no eigenvalues with positive real parts are available).

3 Self-similar profiles and generic blow-up

Consider the unperturbed rescaled equation (1.14). We begin with the linearized stability analysis and describe invariant subspaces, where, without loss of generality, we consider the symmetric case. The non-symmetric consideration includes more eigenfunctions and is done similarly.

3.1 Invariant eigenspaces

Let us write (1.14) in terms of the linearized operator (2.1) as,

$$g_\tau = (\mathbf{B}^* + I)g + \mathbf{D}(g) \tag{3.1}$$

with the nonlinear operator

$$\mathbf{D}(g) = \mathbf{A}_1(g) - (\mathbf{B}^* + I)g \equiv \beta[(g_y)^3]_y + e^g - (1 + g).$$

On solutions $g(\cdot, \tau) \in H^4_{\rho^*}(\mathbf{R}^N)$ the perturbation is quadratic,

$$\mathbf{D}(g) = \frac{1}{2}g^2 + O\left(\|g\|^3_{\rho^*}\right) \quad \text{as } \|g\|_{\rho^*} \to 0. \tag{3.2}$$

In what follows, we restrict our attention to symmetric in x solutions $u = u(|x|, t)$ and hence to symmetric in y rescaled solutions $\theta = \theta(|y|, \tau)$ and $g = g(|y|, \tau)$. In the space $L^2_{0,\rho^*}(\mathbf{R})$ of symmetric functions, in view of (2.5),

$$\sigma(\mathbf{B}^* + I) = \{\tilde{\lambda}_k = 1 - k/4,\ k = 0, 2, 4, \ldots\}. \tag{3.3}$$

By completeness in Lemma 2, we have that 'tilde' stands for subspace where $\{\psi^*_k\}$ is closed

$$\tilde{L}^2_{0,\rho^*}(\mathbf{R}) = E^u(0) \oplus E^c(0) \oplus E^s(0),$$

where $E^u(0)$, $E^c(0)$ and $E^s(0)$ are the unstable, centre and stable subspaces of $\mathbf{B}^* + I$,

$$E^u(0) = \text{Span}\{\psi^*_0, \psi^*_2\}, \quad E^c(0) = \text{Span}\{\psi^*_4\}, \quad E^s(0) = \text{Span}\{\psi^*_6,\ \psi^*_8, \ldots\}.$$

Consider two one-dimensional unstable subspaces corresponding to positive eigenvalues

$$\tilde{\lambda}_0 = 1,\ \ \psi^*_0(y) = 1 \quad \text{and} \quad \tilde{\lambda}_2 = 1/2,\ \ \psi^*_2(y) = y^2/\sqrt{2}.$$

As is usual in blow-up problems, the first unstable mode with $k = 0$ corresponds to the instability of blow-up behaviour with respect to perturbations of the blow-up time T. The second mode with $k = 2$ then describes an actual instability of the trivial solution $g \equiv 0$ in the space of rescaled solutions having the same fixed blow-up time T. We expect such orbits are uniformly bounded in Q_0, i.e. (1.14) does not admit symmetric global solutions which become unbounded as $\tau \to \infty$.

Consider now the centre subspace $E^c(0)$ corresponding to

$$\tilde{\lambda}_4 = 0 \quad \text{with } \psi^*_4(y) = \frac{1}{\sqrt{24}}(y^4 + 24). \tag{3.4}$$

Let us present a simple calculation showing that the behaviour on the centre manifold is semistable. In view of known spectral and sectorial properties of operators $\mathbf{B}$, $\mathbf{B}^*$ [8, 14], the centre (and stable, see the next section) manifold behaviour can be justified by the stable invariant manifold theory in interpolation spaces (see Lunardi [26], Chapter 9).

Proposition 2 *Let* $g(\cdot,\tau) \in H^4_{0,\rho^*}(\mathbf{R})$ *exhibit the centre subspace dominance, i.e.*

$$g(\cdot,\tau) = a_4(\tau)\psi_4^*(\cdot) + w(\cdot,\tau) \quad \textit{for} \ \ \tau \gg 1, \tag{3.5}$$

where $w(\cdot,\tau) = o(\|g(\cdot,\tau)\|_{\rho^*}) = o(|a_4(\tau)|)$ *as* $\tau \to \infty$. *Then*

$$a_4(\tau) = \frac{1}{\gamma_0 \tau}(1 + o(1)) > 0 \quad \textit{as} \ \ \tau \to \infty, \ \textit{where} \ \ \gamma_0 = -\frac{1}{2}\langle(\psi_4^*)^2, \psi_4\rangle = 68\sqrt{6}. \tag{3.6}$$

It follows from (3.6) that $a_4(\tau)$ cannot be negative in any neighbourhood of $\tau = \infty$ meaning a one-sided (from below) instability of the centre manifold behaviour.

Proof Looking for a solution of (3.1) in the form of eigenfunction expansion

$$g(\cdot,\tau) = \sum a_k(\tau)\psi_k^*(\cdot),$$

we substitute it into (3.1) and multiply by ψ_k in $L^2(\mathbf{R})$ to arrive at the dynamical system for a_k in the form

$$\dot{a}_k = \tilde{\lambda}_k a_k + \langle \mathbf{D}(g), \psi_k \rangle, \quad k = 0, 2, \ldots .$$

Consider the equation for the coefficient a_4 with $\tilde{\lambda}_4 = 0$. In view of assumption (3.5) and (3.2), assuming that $|a_4| \ll 1$, we obtain that

$$\dot{a}_4 = -(\gamma_0 + o(1))a_4^2 \quad \text{for} \ \ \tau \gg 1. \tag{3.7}$$

We calculate $\gamma_0 > 0$ by using $\psi_4 = 24^{-1/2}D^4F$ and the adjoint eigenfunction in (3.4). Integrating (3.7) as a standard ODE, we obtain that any small solution for $\tau \gg 1$ has the asymptotic behaviour (3.6). □

It follows from the quadratic ODE (3.7) that the centre manifold behaviour exhibits a typical semistable 'saddle-node' structure.

3.2 Two self-similar profiles

The actual construction of self-similar profiles leads to a complicated multi-dimensional shooting problem which involves two parameters C, C_1 in the bundle (1.17) and two parameters, say, $\mu = f(0)$ and $\nu = f''(0)$, at the origin. We thus observe a dramatic increase of the topological complexity of the existence problem for higher-order equations, unlike the second-order one where just a single parameter from each side occurs, and moreover, typical Maximum Principle ideas apply in the phase-plane analysis. The rigorous proof of existence of f_1 (and especially of f_2) is a hard open multi-dimensional topological connection problem for the nonlinear ODE (1.16). This problem is non-autonomous and has neither Hamiltonian nor variational structure and is hence not amenable to the

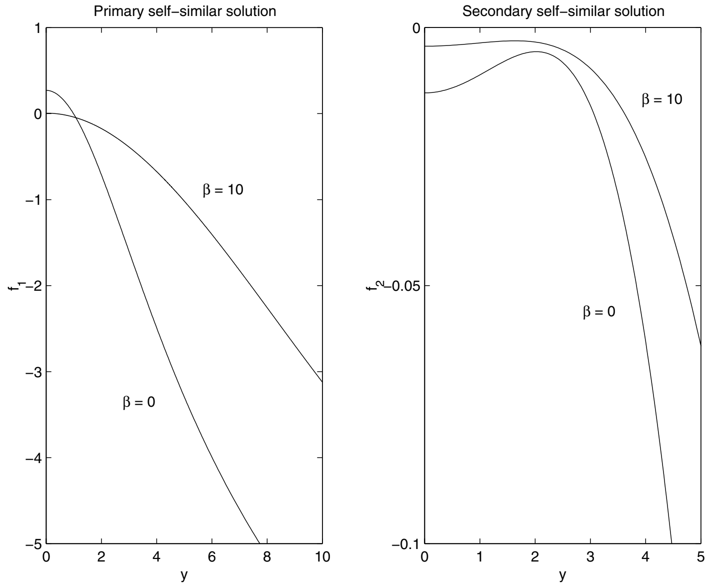

FIGURE 1. Solutions to the ODE describing exact self-similar solutions.

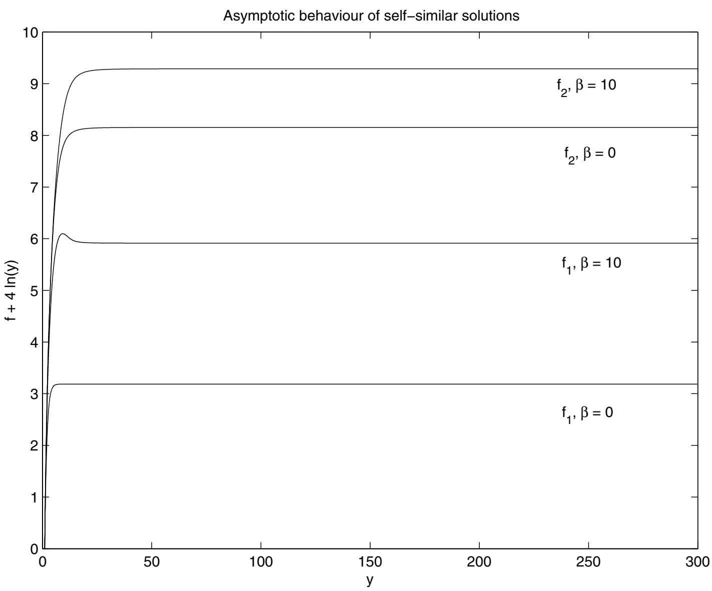

FIGURE 2. Far-field behaviour of solutions to the self-similar ODE.

merthods currently used to prove existence of solutions to fourth-order ODEs. In our further analysis, we rely of carefully chosen numerical methods and extensive numerical experiments.

In Figures 1 and 2 we present the results of numerical solution of the ODE (1.16) for different values of the parameter β. A careful numerical experiment suggests that there exist *exactly* two different solutions $f_1(y)$ and $f_2(y)$ including the case $\beta = 0$, i.e. the fourth-order parabolic equation (1.5) admitting two non-trivial blow-up similarity solutions (1.15). Figure 1 shows the structure local to the origin, while Figure 2 shows the expected far field behaviour. The solutions were obtained using a collocation code which guarantees a residual tolerance [30]. Symmetry conditions were imposed at the origin ($f'(0) = f'''(0) = 0$) and minimal growth was enforced at the far field by setting $f''(L) = f'''(L) = 0$ for various increasing values of L until numerical convergence was observed, typically $L = 2000$ was the final value used. This admits but does not demand the asymptotic behaviour $f \sim -4 \ln y$.

Existence of precisely two similarity profiles can be connected with the above analysis of unstable and centre manifold behaviour. Namely, we conjecture that for the evolution (1.12), there hold

(i) The equilibrium connection $0 \to f_1$ occurs via the second unstable mode with $k = 2$ in (3.3), $\psi_2(y) = y^2/\sqrt{2}$.

(ii) The second similarity profile f_2 occurs in the orbit connection $0 \to f_2$ as the result of the unstable centre manifold behaviour (see Proposition 2) corresponding to $k = 4$ and the eigenfunction in (3.4). Since according to (3.5), this is instability from below on strictly negative orbits $g(y, \tau) < 0$ with $a_4(\tau) < 0$, such an orbital connection leads to a strictly negative second profile $f_2(y) < 0$, which is seen in Figure 1.

These exhaust the unstable modes and possible orbital connections. Basic properties of connecting equilibria and transversality of intersections of the corresponding stable and unstable manifolds are known for the one-dimensional second-order parabolic equations

$$u_t = u_{xx} + f(x, u) \quad \text{in } (0,1) \times \mathbf{R}_+, \quad u = 0 \text{ at } x = 0, 1 \text{ for } t > 0. \tag{3.8}$$

See Henry [19], Angenent [1] and Chen *et al.* [7], where the results are obtained by using Sturm's Theorem for the nonincrease of the number of zeros (intersection) of solutions to linear second-order parabolic equations. This Sturmian property is not true for the fourth and higher-order parabolic equations and in general the structure of connecting orbits remains an open problem.

Let us state the following conjecture motivated by a number of numerical experiments and the preceding analysis.

Conjecture 1 *For any* $\beta \geqslant 0$, *the problem* (1.16), *with operator* (1.12) *has precisely two solutions* f_1 *and* f_2.

It is worth mentioning that the general higher-order semilinear parabolic equation with a power nonlinearity

$$u_t = (-1)^{m+1} D_x^{2m} u + |u|^{p-1} u, \quad p > 1, m > 1, \tag{3.9}$$

also admits two nontrivial self-similar blow-up solutions for $m = 2$ [4]. We should also note that the linear analysis suggests the existence of two solutions and is, on it's own, simply a lower bound. However, through extensive numerical experiments no additional solutions have been found. Let us also mention that the so-called μ—bifurcation diagram [4], applied to the ODE (1.16), where the coefficient of the non-autonomous term in the

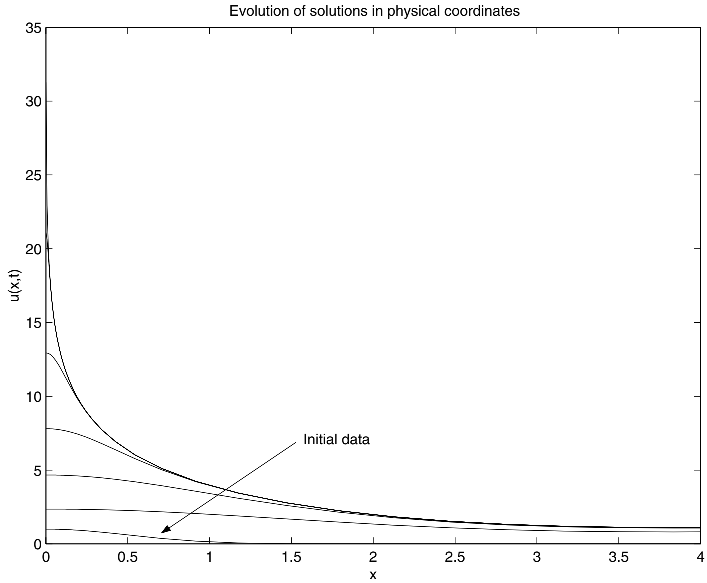

FIGURE 3. Finite time blow-up.

linear operator, $\frac{1}{4}yD_y$, is replaced by a parameter $\mu \in \mathbf{R}$ to give $\mu y D_y$. Analysis of this new bifurcation problem also suggests the existence of precisely two solution branches at $\mu = 1/4$ (though the global bifurcation structure is not rigorously proved). This is in agreement with Budd *et al.* [4], where additional solutions to those predicted by the linear analysis were found only for $m \geqslant 3$.

3.3 Generic blow-up

We conjecture that the first similarity profile $f_1(y)$ describes the *stable* generic blow-up for general solutions of unperturbed equation (1.4) and the perturbed original one (1.9). In Figure 3 we show typical profiles of the evolution, showing single-point blow-up at the origin and formation of the final time profile. Because the blow-up time T is not known, a-priori we cannot immediately compare the solutions to the expected asymptotic profiles. To reconstruct the convergence onto the self-similar solution from time integrations of the PDEs (1.4) and (1.3), we rescale under (1.11) in the following way. At each time t, we define $\lambda = \exp(u(0,t))$ and set

$$\theta(y,\tau) = -\ln\lambda + u\big(x\lambda^{1/4}, t\big) \quad \text{where } \tau = 1/\lambda.$$

Figure 4 presents a typical example of convergence of the rescaled solution $\theta(y,\tau)$ (shown with solid lines) to the first similarity profile $f_1(y)$ (shown with a dashed line). Under the rescaling, the profiles are defined on ever larger ranges of $y = x/(T-t)^{1/4}$.

The numerical simulations of the PDEs were done using collocation on a moving grid [22, 34]. This method moves the grid points to track the solution so as to follow any scalings in the problem. This allows integration of problems whose scale changes

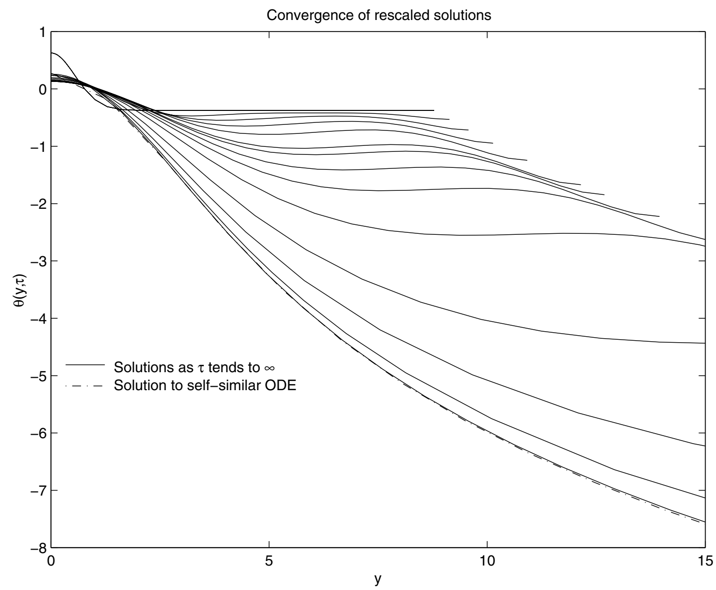

FIGURE 4. Convergence of rescaled profiles.

dramatically over the time domain of interest. In this problem, while the computed solution is never large, it is localized at the origin and care must be taken to resolve this fine structure. In the final blow-up limit ($u(0, T^-) = 32, e^u \sim 10^{13}$), the grid spacing varies over five orders of magnitude using only 20 grid points. Both full periodic and Neumann boundary conditions were implemented numerically and they gave the same blow-up dynamics.

A rigorous justification of a 'stability' of the first similarity profile f_1 is a hard open problem. It can be shown that the rescaled PDE (1.13) *is not* a gradient system (unlike typical cases occurring for second-order semilinear heat equations), so that we cannot expect any 'non-local' stability analysis to apply. On the other hand, the application of the principle of linearized stability (e.g. see Lunardi [26]) leads to a non self-adjoint differential operator with non-constant coefficients and unknown discrete spectrum (possibly, to be calculated by suitable numerical methods).

4 Countable family of other blow-up patterns

We now identify other types of blow-up patterns, which are not stable but can occur in the present explosion-convection problem creating a special singularity formation phenomenon and a discrete 'spectrum' of final-time profiles. For equations (1.4) and (1.9), we construct such blow-up patterns generated by the invariant manifold analysis associated with the centre and stable subspaces of the linearized operator $\mathbf{B}^* + I$ in (3.1). It turns out that such an analysis can be done similarly to that for the semilinear equation with the power nonlinearity, (3.9), for arbitrary order $2m \geqslant 4$ [14].

4.1 Centre manifold pattern: solutions taking infinite values on moving boundaries

By virtue of Proposition 2 and assuming the asymptotic behaviour (3.6), on any compact subset $\{|y| \leqslant c\}$, $c > 0$, the rescaled solution has the form

$$g(y, \tau) = \frac{1}{\gamma_0 \tau} \frac{1}{\sqrt{24}}(y^4 + 24) + o\left(\frac{1}{\tau}\right) > 0 \quad \text{as } \tau \to \infty. \tag{4.1}$$

We then introduce the new rescaled variable

$$\zeta = y/\tau^{1/4} \equiv x/[(T-t)|\ln(T-t)|]^{1/4}, \tag{4.2}$$

so that (4.1) implies that as $\tau \to \infty$,

$$g = \gamma_1 \zeta^{2m}(1 + o(1)) > 0 \quad \text{for small } \zeta > 0, \quad \gamma_1 = 1/\sqrt{24}\gamma_0 = 1/816. \tag{4.3}$$

This extra rescaling forms a remarkable spatial variable (4.2) with an additional logarithmic factor. For the second-order heat equations like (1.1), a similar variable occurs with the exponent 1/2 instead of 1/4. The idea of such non scaling invariant hot-spot variable goes back to the beginning of the 1970s [21], where the equation $u_t = u_{xx} + u^3$ was studied.

Applying the scaling (4.2) in (1.14), we deduce that $g = g(\zeta, \tau)$ satisfies the following perturbed equation:

$$g_\tau = \mathbf{H}_4(g) + \frac{1}{\tau}\mathbf{F}(g) \quad \text{for } \tau > \tau_0, \tag{4.4}$$

with the first-order operator

$$\mathbf{H}_4(g) = -\frac{1}{4}\zeta g_\zeta + e^g - 1, \quad \text{and} \quad \mathbf{F}(g) = -g_{\zeta\zeta\zeta\zeta} + \beta[(g_\zeta)^3]_\zeta - \frac{1}{4}\zeta g_\zeta.$$

As $\tau \to \infty$, (4.4) is an asymptotically small singular perturbation of order $O(1/\tau)$ (it is of crucial importance that the perturbation is not integrable, $1/\tau \notin L^1((1, \infty))$) of the first-order Hamilton–Jacobi equation

$$h_\tau = \mathbf{H}_4(h). \tag{4.5}$$

Such singularly perturbed dynamical systems occur in several reaction-diffusion equations (see Galaktionov & Vazquez [15], where a general stability theorem is available). The main hypothesis of this stability approach is the uniform Lyapunov stability of the ω-limit set of the unperturbed equation (4.5) established in Galaktionov & Vazquez [16] for general equations like (4.5) in a Banach space $C_\rho(\mathbf{R}_+)$ with a singular weight. On the other hand, for such higher-order parabolic problems, compactness of rescaled orbits is a difficult problem and the analysis below is formal.

Assuming that we can pass to the limit in the singularly perturbed equation (4.4), we have that the orbit approaches the stationary profiles satisfying

$$\mathbf{H}_4(f) = 0 \quad \text{for } \zeta > 0, \quad f(0) = 0.$$

Integrating this ODE in the class of nonnegative functions $f \geqslant 0$, which is one of the matching conditions due to the positivity of expansions (4.1) and (4.3), we obtain a one-parameter family of the limit profiles,

$$f(\zeta) = -\ln(1 - A\zeta^4) \quad \text{with a parameter } A \geqslant 0. \tag{4.6}$$

Since $f(\zeta) = A\zeta^4 + O(\zeta^8)$ as $\zeta \to 0$, comparing with (4.3) for the intermediate values of ζ yields the unique stable profile (4.6), $f_*(\zeta)$, with the constant $A = \gamma_1$.

Thus, in terms of the new rescaled variable (4.2) with an extra logarithmic factor, the centre manifold behaviour is governed by a unique stationary solution of the Hamilton–Jacobi equation (4.5):

$$f_*(\zeta) = -\ln(1 - \gamma_1\zeta^4).$$

This means that in the original variables (x, t), the corresponding blow-up pattern takes the asymptotic form

$$u_4(x,t) = -\ln(T-t) - \ln(1 - \zeta^4/816) + \ldots, \quad \zeta = x/[(T-t)|\ln(T-t)|]^{1/4},$$

i.e. it blows up as $t \to T$ in a shrinking domain $\{|x| \leqslant 4\sqrt{3}[(T-t)|\ln(T-t)|]^{1/4}\}$ and $u_* = +\infty$ on its lateral boundary. Such an infinite Dirichlet condition on moving boundaries is a typical feature for fourth-order PDEs like (1.3), (1.9) or (1.5) (unlike the second-order heat equation (1.1) where such initial-boundary value problems are not locally solvable in general in natural functional classes). Note that the ODE (1.16) admits solutions $f(y)$ blowing up as $y \to y_0 \pm 0$ for any finite y_0 with the singularity given in the first approximation by the equation $f'''' = e^f$.

4.2 Stable manifold patterns

Let us describe the rest of blow-up patterns associated with stable subspaces of the linearized operator. The patterns on the stable manifold tangent to $E^s(0)$ are generated by the corresponding eigenfunctions with the following asymptotic behaviour of solutions of (3.1) for $\tau \gg 1$: on compact subsets

$$g(y,\tau) = Ce^{(1-k/4)\tau}\psi_k^*(y) + \ldots, \quad k = 6, 8, \ldots \quad (\text{hence } \tilde{\lambda}_k = 1 - k/4 < 0), \tag{4.7}$$

where $C \neq 0$ depends upon the initial data. The adjoint eigenfunctions ψ_k^* given by (2.7) do not change sign. Therefore, as we show, positive C in (4.7) correspond to blow-up patterns in shrinking domains as $t \to T$, while negative values of C generate blow-up patterns which are well-defined in $\mathbf{R} \times (0, T)$.

We concentrate on negative C and set $C \mapsto -C$. Then (4.7) yields for large y as $\tau \to \infty$,

$$g(y,\tau) = -Be^{(1-k/4)\tau}y^k(1 + o(1)) + \ldots \equiv -B\zeta^k + \ldots, \; |\zeta| \ll 1, \; B = C/\sqrt{k!} > 0, \tag{4.8}$$

where ζ is the new spatial variable

$$\zeta = ye^{-\tau(k-4)/4k} \equiv |x|/(T-t)^{1/k}, \quad k = 6, 8, \ldots.$$

Then $g = g(\zeta, \tau)$ satisfies the exponentially perturbed equation

$$g_\tau = \mathbf{H}_k(g) - e^{-(1-4/k)\tau}\left\{-g_{\zeta\zeta\zeta\zeta} + \beta[(g_\zeta)^3]_\zeta\right\}, \tag{4.9}$$

with the Hamilton–Jacobi operator

$$\mathbf{H}_k(g) = -\frac{1}{k}\,\zeta g_\zeta + e^g - 1.$$

Using the same stability arguments, similar to the above centre case $k = 4$, we conclude that, under necessary compactness hypotheses, the orbit $\{g(\cdot, \tau)\}$ approaches the stationary subset of the unperturbed equation (4.9) corresponding to $\tau = \infty$. Solving the ODE $\mathbf{H}_k(f) = 0$ in the class of nonpositive solutions, we obtain the family

$$f(\zeta) = -\ln(1 + A\zeta^k) \quad \text{with a parameter} \quad A \geqslant 0.$$

Matching expansion $f(\zeta) = -A\zeta^k + O(\zeta^{2k})$ as $\zeta \to 0$ with (4.8), we obtain

$$A = B = C/\sqrt{k!}.$$

We thus obtain the following approximate representation of the stable manifold pattern on compact subsets in ζ:

$$u_k(x,t) = -\ln[(T-t)(1 + A\zeta^k)] + \ldots, \quad \zeta = |x|/(T-t)^{1/k}.$$

Passing to the limit $t \to T$, we derive a countable subset of final-time profile created at $t = T$ via such a blow-up evolution

$$u_k(x, T^-) = -k\ln|x| - \ln A + \ldots \quad \text{as } x \to 0, \quad k = 6, 8, \ldots. \tag{4.10}$$

Thus, the final-time profiles *are not* arbitrary and there exists a countable family (4.10) of those which can occur in the fourth-order PDE (1.4).

5 Conclusions

In this paper we have described countable families of blow-up patterns in a model from convection-explosion theory. By numerical methods and analytic estimates we have found that, contrary to results for a number of similar second-order semilinear problems, there exist two self-similar solutions to a reduced model. Moreover, the primary self-similar solution was found to be stable in a rescaled sense in the full equation as well. The existence of these solutions is understood through close inspection of an associated linear operator. Finally, we note that this phenomenon is robust, occurring in many higher-order equations [4].

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