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On very singular similarity solutions of a higher-order semilinear parabolic equation*

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Abstract

We study the large-time behaviour of solutions of a semilinear $2m$th-order parabolic equation

$$u_t = -(-\Delta)^m u - |u|^{p-1}u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \qquad p > 1,$$

with bounded integrable initial data u_0 decaying exponentially at infinity. For the semilinear heat equation ($m = 1$), the asymptotic behaviour was established in detail in the 1980s. Our main goal is to justify that, for any $m \geqslant 1$ in the subcritical range $1 < p < p_0 = 1 + (2m/N)$, there exists a finite number, $M \sim N(p_0 - p)/2(p-1) \to \infty$ as $p \to 1^+$, of different very singular self-similar solutions of the form

$$u_*(x,t) = t^{-1/(p-1)} V(y), \qquad y = \frac{x}{t^{1/2m}},$$

where each V is a radial, exponentially decaying solution of the elliptic equation

$$-(-\Delta)^m V + \frac{1}{2m} \nabla V \cdot y + \frac{1}{p-1} V - |V|^{p-1} V = 0 \quad \text{in } \mathbf{R}^N.$$

By a perturbation technique, we establish the existence of radially symmetric very singular solution profiles V_l for p close to critical bifurcation exponents $p_l = 1 + (2m/(l+N))$, $l = 0, 2, \ldots$, where the first one, V_0, is shown to be stable. Discrete and countable subsets of other self-similar and approximately self-similar patterns are introduced.

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1. Introduction. Very singular similarity solutions

The model semilinear parabolic equation with absorption. We consider the Cauchy problem for the $2m$th-order semilinear parabolic equation

$$u_t = -(-\Delta)^m u - u^p \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \qquad u(x,0) = u_0(x) \in L^\infty(\mathbf{R}^N) \cap L^1(\mathbf{R}^N), \tag{1.1}$$

where Δ is the Laplacian in $\mathbf{R}^N$ and, in most of the cases, the initial data $u_0(x)$ are assumed to decay exponentially fast as $x \to \infty$ (see (2.1)). For convenience, we use the notation

$$u^p := |u|^{p-1}u, \qquad p > 1.$$

The operator on the right-hand side of (1.1) is monotonic, coercive, and affords a unique global weak solution of the Cauchy problem (see Lions's book [34]), which is a bounded classical solution for a wide parameter range (see section 2). We study the behaviour of such global solutions as $t \to \infty$.

From the beginning of the 1980s, the problem of asymptotics for the semilinear heat equation (1.1) with $m = 1$, which became a canonical diffusion–absorption equation (together with the reaction–diffusion one $u_t = \Delta u + u^p$ from combustion theory), led to the study of a new class of similarity solutions called very singular solutions (VSSs). Various asymptotic techniques were developed to prove the existence, uniqueness and global stability of the VSS in the subcritical parameter range $1 < p < p_0 = 1 + (2/N)$ (for non-negative solutions). Interesting new asymptotic phenomena were also discovered in the supercritical range $p > p_0$, and in the critical case $p = p_0$. We refer to the papers [5–8, 17, 23, 29–31] (this list of references is not complete and includes only the papers to be used herein); see also surveys in [40, chapter 2] and [25, chapter 4]. Historically, the term VSS arose because the first solutions discovered for the second-order *semilinear heat equation* posed for $u \geqslant 0$,

$$u_t = \Delta u - u^p \tag{1.2}$$

with a Dirac measure as initial data were quite naturally termed 'singular solutions' but are not as singular as the VSS (for which $u(x,0)$ is not a measure) [29].

The study of VSS-like asymptotics generated numerous barrier, comparison, reflection, Lyapunov and variational techniques, based on the Maximum principle, which were later applied to a wide class of semilinear and quasilinear second-order parabolic equations describing various reaction, diffusion, absorption and convection processes. The VSSs of nonlinear parabolic equations discovered in studying the model equation (1.1), $m = 1$, represented a new important class of stable generic asymptotics of evolution PDEs.

The structure of equation (1.1) dictates that the critical absorption exponent p establishes a certain balance between the elliptic 'diffusivity' operator and the algebraic absorption operator, and this phenomenon is expected to exist for any order $m \geqslant 1$. This is of principal importance in elucidating the properties of higher-order equations similar to those which are well understood for their second-order counterparts. We present the results of asymptotic analysis for the $2m$th-order semilinear equations, where the semigroups are not order-preserving and the positivity of solutions is not an invariant property. We consider general initial data with exponential decay at infinity and classify the asymptotic behaviour of solutions by describing various countable and continuous subsets of special patterns. Our main goal is to show that the general portrait of asymptotic patterns remains the same for any $m \geqslant 1$ including the classical case $m = 1$ for solutions that change sign, where some of our results are new. Indeed, the mathematics of higher-order equations becomes essentially more delicate (various self-adjoint, potential and order-preserving properties of the operators and semigroups involved are lost) and the rigorous justification of some conclusions remains an open problem. Consequently, while our results extend the known behaviour for $m = 1$ to $m > 1$, new methods will be required to fully justify our conclusions, as we will show that the classical methods no longer apply.

On related models with similar VSS-like asymptotics. Let us mention some other models for which our main conclusions and approaches can play a role. Higher-order equations such as (1.1) occur in various physical and mechanical applications. For instance, the extended Fisher–Kolmogorov equation from the theory of bistable systems, pattern formation, wave propagation and phase transition near singular Lifshitz points,

$$u_t = -\gamma u_{xxxx} + \alpha u_{xx} + \beta u - u^3, \tag{1.3}$$

studied in detail in the last decade (see models, main results and references in the book by Peletier and Troy [38]), contains precisely two operators in the limit case $\alpha = \beta = 0$. Then, instead of kinks, pulses, periodic solutions and travelling waves [38], the VSS problems on existence, stability and multiplicity of solutions play the key role. A similar interaction of higher order linear and nonlinear quadratic Hamilton–Jacobi operators can occur in the limit case $\gamma = 0$ of the Kuramoto–Sivashinsky equation from flame front propagation theory (see [38, p 9]):

$$u_t = -u_{xxxx} - \gamma u_{xx} - \tfrac{1}{2}(u_x)^2.$$

Actually, here and in (1.3), for some asymptotic regimes, the lower-order terms can be negligible as $t \to \infty$, producing exponentially small perturbations of the corresponding rescaled equations. VSSs can be introduced for a generalized equation $u_t = -u_{xxxx} - |u_x|^p$ or for a number of similar ones with more general lower-order operators including the modified Kuramoto–Sivashinsky equation describing flame propagation and several other phenomena like solidification of a hyper-cooled melt [2].

Another well-known class of models exhibiting similar asymptotic properties of global solutions is the stable Cahn–Hilliard equation (see [36, 37] and [16] for the main applications and results):

$$u_t = -\Delta^2 u + \Delta u^p. \tag{1.4}$$

Unlike (1.1), this equation is divergent and preserves initial finite mass and momentum. Of course, the conservation laws affect some crucial properties of solutions but nevertheless we claim that there exist other common features of the asymptotics of global solutions of (1.1) and (1.4) (see, for instance, [18]). The main difference is that the conservation laws introduce extra parameters of mass and momentum, which, for some critical exponents p, make the subsets of possible asymptotics to be continuous relative to these parameters. For the non-conservative equation (1.1), these subsets are expected to be discrete always which, possibly, is a generic property of non-conservative equations. We also anticipate that some of our conclusions can be applied to the more complicated quasilinear thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) + \Delta u^p. \tag{1.5}$$

For $n > 0$, this model describes the dynamics of thin films of viscous fluids in the presence of two competing forces, where as usual only non-negative solutions are taken into account (see references in [2]). For $n = 0$, (1.5) reduces to the Cahn–Hilliard equation (1.4), for which the non-negativity property is lost. We expect that (1.5) can admit various branches of the VSSs that change sign. Similar phenomena can be observed for the non-divergent thin film equation with absorption:

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) - u^p, \qquad n \in (0, 3), \quad p > n + 1,$$

where some VSSs can also change sign. Other non-negative solutions of VSS type can be obtained by means of free-boundary problems posed in moving bounded domains rather than the Cauchy problem in $\mathbf{R}^N$.

Choosing the simple model equation (1.1) associated with the classical semilinear heat equation with absorption (1.2), we are going to present some conclusions and approaches to the VSS-like solutions, which can be applied to various nonlinear higher-order parabolic equations where the operators involved have similar structures.

Plan of the paper. The organization of this paper is as follows. In the remainder of this section we introduce self-similar solutions and briefly describe our main results. The next three sections contain some preliminary and auxiliary results. In section 2 we briefly discuss the global existence of classical solutions to (1.1) (not straightforward for $m > 1$). Section 3 is devoted to the asymptotic analysis of the ODE for similarity profiles and section 4 collects known facts about the linear non-self-adjoint operators under consideration for subsequent use. Sections 5–8 present a PDE analysis of (1.1). In section 5 we describe three different types of asymptotic patterns formed, loosely speaking, by the unstable, stable and centre manifold behaviour for the rescaled PDE. In section 6 we consider a related bifurcation problem for the ODE which motivates the estimate on the number of possible solutions. We also prove the existence of a stable VSS profile for $p \approx p_0^-$ and present supporting numerical evidence. In section 7, we show that the rescaled equation is not a gradient system in any natural weighted space, and hence general stability Lyapunov–La Salle's results do not apply, unlike the case $m = 1$. In section 8, for completeness of the asymptotic analysis, we briefly discuss a continuous spectrum of similarity solutions corresponding to data with algebraic (non-exponential) decay at infinity.

1.1. The first critical absorption exponent

We begin by collecting existing results concerning the asymptotic behaviour of global solutions with initial data decaying exponentially at infinity (see (2.1) and a precise functional setting in a weighted L^2 space in section 4). It is known that for equation (1.1)

$$p_0 = 1 + \frac{2m}{N} \tag{1.6}$$

is a *critical* exponent in the following sense. (i) In the supercritical range $p > p_0$, for a class of sufficiently small initial data, the solutions behave, as $t \to \infty$, as the fundamental solution (up to a constant multiplier $C \neq 0$ specified by initial data),

$$b(x,t) = t^{-N/2m} f(y), \qquad y = \frac{x}{t^{1/2m}}, \tag{1.7}$$

of the linear parabolic equation

$$u_t = -(-\Delta)^m u. \tag{1.8}$$

The rescaled kernel f is the unique radial solution of the elliptic equation

$$\mathbf{B} f \equiv -(-\Delta)^m f + \mathcal{L}_0 f = 0 \quad \text{in } \mathbf{R}^N, \qquad \int f = 1; \qquad \mathcal{L}_0 = \frac{1}{2m} y \cdot \nabla + \frac{N}{2m} I, \tag{1.9}$$

satisfying, for some positive constants $D > 1$ and $d > 0$ depending on m and N [14],

$$|f(y)| < DF(y) \equiv D\omega_1 \, \mathrm{e}^{-d|y|^\alpha} \quad \text{in } \mathbf{R}^N, \qquad \alpha = \frac{2m}{2m-1} \in (1,2), \tag{1.10}$$

$\omega_1 > 0$ being a normalization constant such that $\int F = 1$. We refer to [13] in which perturbation techniques are applied to any equation like (1.1) with the lower-order term replaced by $\pm|u|^p$ or $\pm|u|^{p-1}u$. For the reaction–diffusion equations

$$u_t = -(-\Delta)^m u + |u|^p,$$

(1.6) becomes the critical Fujita exponent [13, 21]. (ii) The critical case $p = p_0$ is studied in [22] where it is established that, for some initial data, global solutions have the following logarithmically perturbed fundamental asymptotic behaviour as $t \to \infty$:

$$u(x,t) = \pm C_0 (t \ln t)^{-N/2m} \left[f\left(\frac{x}{t^{1/2m}}\right) + \mathrm{o}(1) \right], \tag{1.11}$$

where the constant $C_0 \neq 0$ depends on m and N but is independent of initial data.

For the semilinear heat equation (1.2), these results were established in the 1980s. In this case $p_0 = 1 + (2/N)$ coincides with the critical Fujita exponent for the reaction–diffusion equation $u_t = \Delta u + u^p$ (see [40, chapter 2] and references therein). Moreover, a complete classification for this semilinear equation is available and in addition to (i) and (ii) above we have the following asymptotic property. (iii) ($m = 1$) In the subcritical range $p \in (1, p_0)$, the asymptotic behaviour of such positive solutions is described by the unique VSS

$$u_*(x,t) = t^{-1/(p-1)} V(y), \qquad y = \frac{x}{t^{1/2}}, \tag{1.12}$$

where $V > 0$ solves a nonlinear ODE (see below, (1.14)). We refer to [6] (an ODE proof of existence of the VSS), [23] (a PDE proof of existence and stability), [31] (uniqueness of the VSS) and [29] (construction of the VSS by monotone approximation of 'very' singular initial data).

1.2. The main result. VSSs in the subcritical range

For the higher-order equation (1.1) with $m \geqslant 2$, we study the existence and multiplicity of similarity solutions and show that in the subcritical range $p \in (1, p_0)$ there exist VSSs of the form (cf (1.12))

$$u_*(x,t) = t^{-1/(p-1)} V(y), \qquad y = \frac{x}{t^{1/2m}}, \tag{1.13}$$

where V is a non-trivial radial solution of the elliptic equation

$$\mathbf{P}_{2m}(V) \equiv \mathbf{B}_1 V - V^p \equiv -(-\Delta)^m V + \frac{1}{2m} \nabla V \cdot y + \frac{1}{p-1} V - V^p = 0 \quad \text{in } \mathbf{R}^N, \tag{1.14}$$

$$V(y) \text{ decays exponentially fast as } |y| \to \infty. \tag{1.15}$$

Condition (1.15) on exponential decay at infinity is most naturally enforced by introducing weighted L^2 and Sobolev spaces (see section 4). The linear part $\mathbf{B}_1$ in equation (1.14) is connected to the operator (1.9) for the rescaled kernel f in (1.7)

$$\mathbf{B}_1 = \mathbf{B} + c_1 I, \quad \text{where } c_1 = \frac{N(p_0 - p)}{2m(p-1)}. \tag{1.16}$$

One of the main goals of this paper is to show by analytical and numerical methods that in the radial setting, the ODE problem (1.14), (1.15) admits at least

$$M(m,p,N) = \sharp_{\text{even}} \left(\frac{N(p_0 - p)}{p-1} \right), \tag{1.17}$$

different non-trivial radial solutions $f = f(|y|)$, where $\sharp_{\text{even}}(z)$ for $z \geqslant 0$ denotes the number of non-negative even numbers $0, 2, 4, \ldots$ not exceeding the integer part $\lfloor z \rfloor$.

We also study the asymptotic stability of the VSS and describe a countable subset of other self-similar or approximately self-similar patterns in the Cauchy problem (1.1). We show that in the supercritical range $p > p_0$, no generically stable (see a precise definition in section 5) non-trivial VSSs exist. On the other hand, there exists an uncountable family of different similarity solutions which are not in L^1 and have special stability properties.

2. Preliminaries. Global existence of classical solutions

According to the kernel estimate (1.10) and a similar one for L^1-VSS profiles in section 3, we consider initial data satisfying, for some constant $k > 0$,

$$u_0(x) = \mathrm{o}(\mathrm{e}^{-k|x|^\alpha}) \qquad \text{as } x \to \infty \quad \left(\alpha = \frac{2m}{2m-1}\right). \tag{2.1}$$

For bounded integrable initial data, *local* in time existence of the classical solution of (1.1) (i.e. of $u(x,t)$ smooth as the regularity of coefficient u^p at $u = 0$ dictates) follows from the equivalent integral equation

$$u(t) = b(t) * u_0 - \int_0^t b(t-s) * u^p(s)\, \mathrm{d}s, \tag{2.2}$$

where $b(t)$ is the fundamental solution (1.7) (see [14, 20] and [42, chapter 15]). For $u_0(x)$ satisfying (2.1), bounded solutions $u(x,t)$ also vanish exponentially fast at infinity according to the asymptotics (1.10) of the fundamental solution (1.7). Namely, by Eidelman's [14, chapter 1] estimates on iterated kernels, the integral operator in (2.2) maps such subsets of continuous functions onto similar (wider) ones (see [13]). For *global* existence of the classical solution, one needs an *a priori* L^∞-bound. It is obvious for $m = 1$ and any p and N, where $\sup_x |u(x,t)| \leqslant \sup_x |u_0(x)|$ by the Maximum Principle, and moreover

$$|u(x,t)| \leqslant [(p-1)t]^{-1/(p-1)} \qquad \text{for all } t > 0 \ (m = 1), \tag{2.3}$$

by comparison with the ODE solution $u' = -u^p$. This is a very strong universal estimate which holds for initial data $u_0 \notin L^1_{\mathrm{loc}}$, $u_0 \geqslant 0$ (a unique *proper* minimal solution $u = \lim u_\varepsilon$ is then defined via classical bounded solutions $u_\varepsilon(x,t)$ with data $u_{0\varepsilon} \uparrow u_0$ pointwise; see [24] on such extensions of order-preserving semigroups). For $m > 1$, the comparison is not valid and estimates like (2.3) are not known (representing an interesting open problem). We begin with some weaker estimates. Multiplying (1.1) by u and u_t in L^2 and integrating by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = -\|\bar{D}^m u(t)\|_2^2 - \|u(t)\|_{p+1}^{p+1} \leqslant 0, \tag{2.4}$$

$$\|u_t(t)\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|\bar{D}^m u(t)\|_2^2 + \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}\right] = 0, \tag{2.5}$$

where $\|\cdot\|_q$ denotes the norm in L^q and $\bar{D}^m = \Delta^{m/2}$ for m even and $\bar{D}^m = \nabla\Delta^{(m-1)/2}$ for m odd. Integrating the second estimate gives

$$\|\bar{D}^m u(t)\|_2 \leqslant C, \quad \|u(t)\|_{p+1} \leqslant C \qquad \text{for } t > 0, \tag{2.6}$$

where $C > 0$ denotes various constants. By classical methods ([34], chapter 1) this makes it possible to establish the global existence of a weak solution, which is unique in view of the monotonicity of the operator on the right-hand side of (1.1). Concerning L^∞ estimates, using the Hölder inequality and (2.6), one can obtain from (2.2) that

$$\begin{aligned} |u(t)| &\leqslant \sup|u_0| \int |b(t)| + \int_0^t \|b(t-s)\|_{p+1}\|u(s)\|_{p+1}^p\, \mathrm{d}s \\ &\leqslant \sup|u_0|\|f\|_1 + C\|f\|_{p+1}\int_0^t (t-s)^{-pN/2m(p+1)}\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

Hence, $u(x,t)$ is bounded in $\mathbf{R}^N \times (0,T)$ for any finite $T > 0$ if the integral in (2.7) converges, i.e. $-Np/(2m(p+1)) > -1$ meaning $p < 2m/(N-2m)_+$. This result is improved by using another Hölder inequality in (2.2)

$$|b(t-s) * u^p(s)| \leqslant U(s)|b(t-s)| * |u(s)|^{p-1} \leqslant U(s)\|b(t-s)\|_{(p+1)/2}\|u(s)\|_{p+1}^{p-1},$$

where $U(s) = \sup_y |u(y, s)|$. This leads to a generalized Gronwall inequality

$$U(t) \leqslant C + C \int_0^t (t - s)^{\beta-1} U(s)\, \mathrm{d}s, \qquad \beta = \frac{N + 2m - p(N - 2m)}{2m(p + 1)}.$$

It implies (see [27], p 188) that, for $\beta > 0$, i.e. in the subcritical Sobolev range

$$p < p_s = \frac{N + 2m}{(N - 2m)_+}. \tag{2.8}$$

$U(t) = \sup_x |u(x, t)|$ is bounded on finite intervals $(0, T)$ and has at most exponential growth at $t \to \infty$. In order to derive a global uniform L^∞ bound we use a scaling technique having various applications to semilinear and quasilinear equations.

Proposition 2.1. *In the range (2.8),*

$$|u(x, t)| \leqslant C \qquad \text{in } \mathbf{R}^N \times \mathbf{R}_+. \tag{2.9}$$

Proof. Arguing by contradiction, we assume that there exist monotone sequences $\{t_k\} \to \infty$, $\{C_k\} \to \infty$ and $\{x_k\} \subset \mathbf{R}^N$ such that

$$\sup_{\mathbf{R}^N \times (0,t_k)} |u(x, t_k)| = C_k \equiv |u(x_k, t_k)|. \tag{2.10}$$

With a sequence $\{a_k\} \to 0^+$ to be determined, we perform the scaling

$$u_k(x, t) \equiv u(x_k + x, t_k + t) = C_k v_k(y, s), \qquad x = a_k y, \qquad t = a_k^{2m} s \tag{2.11}$$

and obtain the following Cauchy problem for v_k:

$$v_s = -(-\Delta)^m v - \delta_k v^p, \qquad \delta_k = a_k^{2m} C_k^{p-1}; \qquad v_{0k} = C_k^{-1} u(t_k), \tag{2.12}$$

where

$$\sup |v_{0k}(y)| = 1, \qquad \|v_{0k}\|_{p+1}^{p+1} = (a_k^N C_k^{p+1})^{-1} \|u(t_k)\|_{p+1}^{p+1}. \tag{2.13}$$

We now set $a_k = C_k^{-(p+1)/N}$ so that $\|v_{0k}\|_{p+1} = \|u(t_k)\|_{p+1} \leqslant C$ and

$$\delta_k = C_k^{\mu} \qquad \text{with } \mu = \frac{(N - 2m)(p - p_s)}{N}. \tag{2.14}$$

Since $\mu < 0$ for $p < p_s$, we have $\delta_k \to 0$ as $k \to \infty$. Fixing a large $s_0 > 0$ and setting $\tilde{v}_k(s) = v_k(s - 2s_0)$, we have that $|\tilde{v}_k(s)| \leqslant 1$ on $(0, 2s_0)$ are uniformly bounded classical solutions of the uniformly parabolic equation (2.12). By the regularity theory [14,20] we have that along a subsequence, $\tilde{v}_k(s) \to \bar{v}_k(s)$ uniformly on compact subsets from $\mathbf{R}^N \times [s_0, 2s_0]$. Passing to the limit in equation (2.12) yields that $\bar{v}(s)$ is a bounded weak solution and hence the classical solution of the Cauchy problem for the linear parabolic equation (1.8) with initial data satisfying $|\bar{v}_0| \leqslant 1$ and $\|\bar{v}_0\|_{p+1} \leqslant C$. By the Hölder inequality, it follows from the convolution $\bar{v}(s) = b(s) * \bar{v}_0$ that

$$|\bar{v}(2s_0)| \leqslant \|b(2s_0)\|_q \|\bar{v}_0\|_{p+1} \leqslant C(2s_0)^{N(1-q)/2mq} \|f\|_q \ll 1, \qquad q = \frac{p + 1}{p},$$

provided that $s_0 \gg 1$, i.e. uniformly in such initial data $\bar{v}_0$. Therefore, $|\tilde{v}_k(y, 2s_0)| \ll 1$ uniformly for $k \gg 1$ contradicting the assumption that $\sup_y |\tilde{v}_k(y, 2s_0)| = 1$. □

It would be interesting to know if the uniform bound (2.9) exists for $p \geqslant p_s$.

3. Preliminaries. The exponential bundle in the ODE as $y \to \infty$

In this section, we describe asymptotics of radial solutions of the ODE (1.14) such that $V(y) \to 0$ as $y \to +\infty$, where y now denotes the radial variable $|y| \geqslant 0$. The linearization of (1.14) about $V = 0$ gives

$$\mathbf{B}_1 V = 0 \quad \text{for } y > 0. \tag{3.1}$$

On such decaying solutions, (1.14) is an asymptotically small perturbation of the linear equation (3.1). The asymptotic analysis of such solutions of perturbed higher-order ODEs such as (3.1) is well established (see [9, chapters III–V] and general asymptotic methods in [19]). According to [9], we first derive the leading, constant coefficient differential operator which defines the asymptotic behaviour. Beginning with the ODE (3.1),

$$(-1)^{m+1}\left(V^{(2m)} + \frac{m(N-1)}{y}V^{(2m-1)} + \cdots\right) + \frac{1}{2m}V'y + \frac{1}{p-1}V = 0, \tag{3.2}$$

to make the balance between the $2m$th derivative and the term $(1/2m)V'y$ clear we set $z = y^\alpha$ with $\alpha = 2m/(2m-1)$ giving the following equation:

$$V^{(2m)} - a_1 V' - z^{-1}a_2 V + z^{-1}\mathbf{C}(z)V = 0. \tag{3.3}$$

Here, $a_1 = (-1)^m(1/2m)\alpha^{1-2m}$, $a_2 = (-1)^m(1/(p-1))\alpha^{-2m}$ and $\mathbf{C}(z)V = \sum_{j=1}^{2m-1}\gamma_j z^{j+1-2m}V^{(j)}$ is a linear operator with bounded coefficients as $z \to \infty$. In this sum, the final coefficient of the highest derivative term, $V^{(2m-1)}$, is $\gamma_{2m-1} = (-1)^{m+1}(1/2m)[1 + m(N-1)(2m-1)]$ and the coefficient of the first derivative, V', is of order $\mathrm{O}(z^{2-2m}) = \mathrm{o}(z^{-1})$ as $z \to \infty$. By the perturbation theory of higher-order linear ODEs, the leading terms of exponentially decaying solutions are described via those for the operator in (3.3) with constant coefficients:

$$V^{(2m)} - a_1 V' = 0. \tag{3.4}$$

Setting $V = \mathrm{e}^{\mu z}$, $\mu \neq 0$, gives the characteristic equation $\mu^{2m} - a_1\mu = 0$, whence

$$\mu^{2m-1} = a_1 = (-1)^m\frac{1}{2m\alpha^{2m-1}} \equiv \rho_0^{2m-1}(-1)^m, \quad \text{with } \rho_0 > 0. \tag{3.5}$$

For any $m \geqslant 1$ there exist $2m-1$ roots $\{\mu_0, \mu_1, \ldots, \mu_{2m-2}\}$ given by

$$\mu_k = \rho_0\,\mathrm{e}^{\mathrm{i}(2k+1)\pi/(2m-1)}, \quad m = 2l+1; \qquad \mu_k = \rho_0\,\mathrm{e}^{\mathrm{i}2\pi k/(2m-1)}, \quad m = 2l, \tag{3.6}$$

where m roots have negative real parts, $\mathrm{Re}\,\mu_k < 0$. These correspond to $k = l, l+1, \ldots, 3l$ for odd $m = 2l+1$ and $k = l, l+1, \ldots, 3l-1$ for even $m = 2l$. Bearing in mind that, for odd m, the root for $k = m$ is real, $\mu_m = -\rho_0$, and for any complex root the corresponding subspace of solutions is two dimensional, we conclude that as $y \to \infty$, there exists an m-dimensional bundle of exponentially decaying solutions. For the second-order case $m = 1$, the bundle is just one dimensional, making it possible to use a phase-plane analysis and shooting arguments to prove the existence of the VSS [6], or apply a monotone parabolic method via simple super- and sub-solutions of the PDE [23]. Obviously, in the presence of the multi-dimensional exponential bundle, the topology of the shooting-type existence analysis becomes more involved and difficult and will not be used later on. In addition, (3.3) admits a solution corresponding to the characteristic root $\mu = 0$ with algebraic decay as $z \to \infty$ described by the first-order operator

$$-a_1 V' - z^{-1}a_2 V = 0 \implies V(z) = Cz^{-(2m-1)/(p-1)}.$$

The existence of solutions with such decay for the perturbed equation (3.3) is established by a standard expansion via a Kummer series converging uniformly for $z \gg 1$. For the linearized equation (3.1) we obtain the algebraic asymptotic behaviour,

$$V(y) = C|y|^{-2m/(p-1)}(1 + \mathrm{o}(1)) \qquad \text{as } y \to \infty, \quad \text{with any } C \neq 0. \tag{3.7}$$

Such solutions do not satisfy condition (1.15) and represent another family of asymptotic similarity patterns for the PDE (1.1) to be discussed in section 8. Summarizing the asymptotic ODE analysis, we have that if V is a VSS profile satisfying problem (1.14), (1.15) with far field behaviour from the exponentially decaying bundle, then the following global estimate holds:

$$|V(y)| \leqslant D_1\, \mathrm{e}^{-d_1|y|^\alpha} \quad \text{in } \mathbf{R}^N, \quad D_1, d_1 > 0. \tag{3.8}$$

Passing to the limit $t \to 0^+$ in (1.13), it follows that such VSSs satisfy

$$u_*(x,t) \to 0 \quad \text{for } x \neq 0 \quad \text{and} \quad |u_*(x,t)|^\beta \to \text{const}\, \delta(x), \qquad \beta = \frac{(p-1)N}{2m}, \tag{3.9}$$

in the sense of bounded measures in $\mathbf{R}^N$. Solutions with algebraic decay (3.7) form the following initial data with the uniform convergence on $[\delta, \infty)$, $\delta > 0$:

$$u_*(x, 0^+) = C|x|^{-2m/(p-1)}. \tag{3.10}$$

4. Spectral properties of linear operators involved

As (1.16) suggests, in order to study the VSS, we need the spectral properties of $\mathbf{B}$ and the corresponding adjoint operator $\mathbf{B}^*$, which will play a role in the further asymptotic analysis of the nonlinear PDE. These are posed in weighted L^2-spaces with the weight functions induced by the exponential estimate of the rescaled kernel (1.10).

4.1. The point spectrum of the non-self-adjoint operator B

For any $m > 1$, $\mathbf{B}$ is not symmetric and does not admit a self-adjoint extension. We consider $\mathbf{B}$ in the weighted space $L^2_\rho(\mathbf{R}^N)$ with the exponentially growing weight function

$$\rho(y) = \mathrm{e}^{a|y|^\alpha} > 0 \quad \text{in } \mathbf{R}^N, \tag{4.1}$$

where $a \in (0, 2d)$ is a sufficiently small constant. Let us ascribe to $\mathbf{B}$ the domain $H^{2m}_\rho(\mathbf{R}^N)$, which is a Hilbert space with the norm

$$\|v\|^2 = \int \rho(y) \sum_{k=0}^{2m} |D^k v(y)|^2 \,\mathrm{d}y,$$

induced by the corresponding inner product. We have $H^{2m}_\rho \subset L^2_\rho \subset L^2$. The spectral properties $\mathbf{B}$ are as follows [13].

Lemma 4.1. *(i)* $\mathbf{B} : H^{2m}_\rho \to L^2_\rho$ *is a bounded linear operator with the real point spectrum*

$$\sigma(\mathbf{B}) = \left\{ \lambda_l = -\frac{l}{2m},\ l = 0, 1, 2, \ldots \right\}. \tag{4.2}$$

The eigenvalues λ_l *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y) = \frac{(-1)^{|\beta|}}{\sqrt{\beta!}}\, D^\beta f(y), \quad \textit{with any } |\beta| = l. \tag{4.3}$$

(ii) The set of eigenfunctions $\Phi = \{\psi_\beta,\ |\beta| = 0, 1, 2, \ldots\}$ *is complete in* L^2_ρ.

In the classical second-order case $m = 1$, $f(y) = (4\pi)^{-N/2}e^{-|y|^2/4}$ is the rescaled positive Gaussian kernel and the eigenfunctions are $\psi_\beta(y) = e^{-|y|^2/4}H_\beta(y)$, where H_β are Hermite polynomials in $\mathbf{R}^N$ [4]. The operator $\mathbf{B}$, with the domain H^2_ρ and weight $\rho = e^{|y|^2/4}$, is self-adjoint and the eigenfunctions form an orthogonal basis in L^2_ρ. Lemma 4.1 gives the centre and stable subspaces of $\mathbf{B}$, $E^c = \mathrm{Span}\{\psi_0 = f\}$, $E^s = \mathrm{Span}\{\psi_\beta, |\beta| > 0\}$.

4.2. The polynomial eigenfunctions of the adjoint operator $\mathbf{B}^*$

Consider the adjoint operator to $\mathbf{B}$,

$$\mathbf{B}^* = -(-\Delta)^m - \frac{1}{2m}y \cdot \nabla. \tag{4.4}$$

For $m = 1$, $\mathbf{B}^* \equiv \frac{1}{\rho^*}\nabla \cdot (\rho^*\nabla)$, $\mathcal{D}(\mathbf{B}^*) = H^2_{\rho^*}$, with weight $\rho^*(y) = e^{-|y|^2/4}$, is self-adjoint in $L^2_{\rho^*}$ and has a discrete spectrum. The eigenfunctions form an orthonormal basis in $L^2_{\rho^*}$ and the classical Hilbert–Schmidt theory applies [4]. For $m > 1$, we consider $\mathbf{B}^*$ in $L^2_{\rho^*}$ with the exponentially decaying weight function

$$\rho^*(y) = \frac{1}{\rho(y)} \equiv e^{-a|y|^\alpha} > 0.$$

Lemma 4.2. *(i)* $\mathbf{B}^* : H^{2m}_{\rho^*} \to L^2_{\rho^*}$ *is a bounded linear operator with the same spectrum as* $\mathbf{B}$, *(4.2). The eigenfunctions* $\psi^*_\beta(y)$ *with* $|\beta| = l$ *are* l*th-order polynomials*

$$\psi^*_\beta(y) = \frac{1}{\sqrt{\beta!}}\left[y^\beta + \sum_{j=1}^{\lfloor|\beta|/2m\rfloor}\frac{1}{j!}(-\Delta)^{mj}y^\beta\right]. \tag{4.5}$$

(ii) The subset $\{\psi^*_\beta\}$ *is complete in* $L^2_{\rho^*}$.

From this definition of the adjoint eigenfunctions, the orthonormality condition holds

$$\langle\psi_\beta, \psi^*_\gamma\rangle = \delta_{\beta,\gamma}, \tag{4.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard L^2 inner product. For $m = 1$, both (4.2) and (4.5) are well-known properties of the separable Hermite polynomials generated by a self-adjoint Sturm–Liouville problem [4].

5. Stability analysis and three types of asymptotic patterns

With the spectral properties of $\mathbf{B}$ in hand we now present arguments on the number of VSS solutions arising from instabilities in the linearization of the PDE about the zero solution.

5.1. Stability of the zero solution

Following (1.13), we use the similarity scaling

$$u = (1+t)^{-1/(p-1)}v, \qquad y = \frac{x}{(1+t)^{1/2m}}, \qquad \tau = \ln(1+t) : \mathbf{R}_+ \to \mathbf{R}_+. \tag{5.1}$$

The rescaled solution $v = v(y,\tau)$ solves the autonomous equation

$$v_\tau = \mathbf{P}_{2m}(v) \equiv \mathbf{B}_1 v - v^p \qquad \text{for } \tau > 0, \quad v(y,0) = v_0(y) \equiv u_0(y). \tag{5.2}$$

The VSS profiles satisfying (1.14), (1.15) are its stationary solutions. We show that at $p = p_0$ the trivial stationary solution $v \equiv 0$ changes its stability, which is a crucial characterization of the critical exponent. As is well known in the general stability and bifurcation theory [10,33], often, this means that $p = p_0$ is a bifurcation point of equilibria (a result to be proved in section 6).

Proposition 5.1. *The trivial solution* $v \equiv 0$ *of equation (5.2) is unstable for* $p \in (1, p_0)$*, and is stable for* $p > p_0$.

Proof. It follows from (1.16) and (4.2) that the linearized (about $v \equiv 0$) operator $\mathbf{B}_1$ in (5.2) has the discrete spectrum

$$\sigma(\mathbf{B}_1) = \left\{ \nu_l = c_1 - \frac{l}{2m},\ l = 0, 1, 2, \ldots \right\}, \tag{5.3}$$

so that $\nu_0 > 0$ for $p \in (1, p_0)$ (for which $c_1 > 0$) and $\nu_0 < 0$ for $p > p_0$ (when $c_1 < 0$). In view of the known spectral properties of $\mathbf{B}$ (see lemma 4.1 and [13]), this stability/instability result in $X = H^{2m}_\rho \cap L^\infty$ follows from the principle of linearized stability (see, e.g. [35, chapter 9]). □

5.2. Non-existence of generically stable VSS for $p > p_0$

In the next section we show that the stationary problem (1.14), (1.15) can admit an arbitrarily large number of different non-trivial solutions $\{V_k,\ k = 0, 1, 2, \ldots, M\}$ (an estimate of M is given below) together with the zero solution denoted by $V^0 \equiv 0$. In order to choose the 'most' stable one, for each profile V_k, we introduce the stable manifold $W^s(V_k)$ of finite codimension and a finite-dimensional unstable one $W^u(V_k)$ associated with stable, centre and unstable subspaces of the linearized operator

$$\mathbf{D}_k = \mathbf{B}_1 - p|V_k|^{p-1} I \tag{5.4}$$

having a discrete spectrum (see section 6).

Definition. *We say that* V_0 *is generically stable in* X *if: (i)* $W^u(V_0) = \emptyset$*, (ii)* $W^u(V_k) \neq \emptyset$ *for all* $k = 1, 2, \ldots, M$*, and (iii) for any* $v_0 \notin (\bigcup_{k=1}^{M} W^s(V_k)) \cup W^s(0)$, $v(\tau) \to V_0$ *as* $\tau \to \infty$*. Note that (iii) includes the global property of stabilization to a stationary solution in the dynamical system (5.2) on* X*, which is not known for general orbits since (5.2) is not a gradient system (see section 7). Therefore, we mainly study the usual (local) stability properties of the VSS for* $p < p_0$ *(section 6). The above definition is currently used to establish a 'weak' non-existence result for* $p > p_0$.

Corollary 5.2. *For* $p > p_0$*, a non-trivial generically stable VSS satisfying (1.14), (1.15) does not exist.*

Indeed, according to proposition 5.1, for $p > p_0$, the trivial profile $V^0 \equiv 0$ is locally stable. Actually, this means that a connection $\{0\} \to \{V_0\}$ in X described by (5.2) does not exist. The same non-existence result remains true in the critical case $p = p_0$. However, in what follows, we will present results on a centre manifold analysis showing that 0 is still stable at the critical exponent. In view of the rather complicated bifurcation structure for equation (1.14), which includes an arbitrarily large number of branches for $p \approx 1^+$ (section 6), the proof of actual non-existence of VSSs for $p \geqslant p_0$ and arbitrary m and N is a difficult problem. On the other hand, the following non-existence result is straightforward, but, we do not believe it to be sharp.

Proposition 5.3. *A non-trivial VSS does not exist for any* $p \geqslant p_* = 1 + (4m/N)$.

Proof. Multiplying (1.14) by V in L^2 yields $-\|\bar{D}^m V\|_2^2 + \mu_* \|V\|_2^2 - \|V\|_{p+1}^{p+1} = 0$, where $\mu_* = 1/(p-1) - (N/4m) \leqslant 0$ for $p \geqslant p_*$. □

5.3. A first estimate on the number of VSSs for $p < p_0$

It follows from (5.3) that in the subcritical case $p \in (1, p_0)$ there exists a finite number of unstable modes corresponding to the trivial equilibrium $V^0 \equiv 0$ of (5.2). This Morse index of $\mathbf{B}_1$ is given by the cardinal number

$$M(m, p, N) = \sharp \left\{ \beta : |\beta| < 2mc_1 = \frac{N(p_0 - p)}{p - 1} \right\}. \tag{5.5}$$

Operator $\mathbf{B}_1 v - v^p$ is known to be potential for $m = 1$ only (see [23, 17, 7]) and equation (5.2) is not a gradient system for $m > 1$ (section 7). The general properties of orbital connections for semilinear higher-order parabolic equations are unknown (see [38, chapter 6] for some particular results). For $m = 1$, a complete classification [1, 28] is based on Sturm's Theorem on zeros associated with the Maximum principle. Notice that Sturm's zero properties of linear and nonlinear eigenfunctions remain valid for some higher-order self-adjoint positive operators [15, 39] (but such zero properties are not supported by the corresponding evolution equations). For the present problem, not variational with no positive self-adjoint operators involved, we cannot guarantee that each unstable mode generates stabilization to a non-trivial stationary profile (unlike the case $m = 1$ where this is actually true). Nevertheless, completing this discussion, we expect that the number (5.5) can characterize the total finite number of non-trivial stationary solutions of the problem (1.14), (1.15), which are 'nonlinear' asymptotic patterns in the PDE under consideration to be compared with the 'linearized' ones presented below. This number of nonlinear patterns increases without bound as p decreases:

$$M(m, p, N) \to \infty \qquad \text{as } p \to 1^+. \tag{5.6}$$

In the radial setting, (5.5) coincides with (1.17). The critical (bifurcation) exponents, at which $M(m, p, N)$ is discontinuous, are given by

$$c_1 - \frac{l}{2m} = 0 \implies p_l = 1 + \frac{2m}{l + N}, \qquad l = 0, 1, 2, \ldots. \tag{5.7}$$

Therefore, (1.6) is the first, maximal exponent corresponding to $l = 0$.

5.4. Stable manifold behaviour: a countable subset of linearized patterns

Unlike the nonlinear VSS patterns, the stable infinite-dimensional subspace $E^s = \text{Span}\{\psi_\beta, \nu_{|\beta|} < 0\}$ of $\mathbf{B}_1$ generates linearized patterns decaying exponentially fast as $\tau \to \infty$. In view of the completeness and orthonormality of eigenfunctions of $\mathbf{B}_1$, for initial data $v_0 \in X$ we use the uniformly converging eigenfunction expansion of solutions that are sufficiently smooth by the parabolic regularity theory [14, 20]

$$v(\tau) = \sum a_\beta(\tau)\psi_\beta. \tag{5.8}$$

The expansion coefficients satisfy the dynamical system

$$a'_\beta = \nu_{|\beta|} a_\beta - \langle v^p, \psi^*_\beta \rangle \qquad \text{for any } \beta. \tag{5.9}$$

The diagonally dominant structure of the system (5.9) shows that if the nonlinear term v^p forms an exponentially decaying perturbation as $\tau \to \infty$, then there exist patterns with exponential decay as $\tau \to \infty$:

$$v(y, \tau) = C\, \mathrm{e}^{\nu_{|\beta|}\tau}(\psi_\beta(y) + \mathrm{o}(1)), \qquad C = C(u_0) \neq 0, \tag{5.10}$$

where ψ_β is a suitable eigenfunction with $\nu_{|\beta|} < 0$. In the asymptotic sense, these are exponentially decaying solutions of the linear equation $v_\tau = \mathbf{B}v$, and such results are well known in linear perturbation theory [10, 20].

5.5. *Centre manifold behaviour at $p = p_l$: logarithmic scaling factors*

It follows from (5.7) that at any $p = p_l$, $\mathbf{B}_1$ has a non-trivial centre subspace. We describe the corresponding asymptotic pattern behaviour in the radial geometry (cf [22] and [13]). For arbitrary even $l = 2, 4, \ldots$, the operator $\mathbf{B}_1$ has the simple eigenvalue 0 with a one-dimensional centre subspace $E^c = \mathrm{Span}\{\psi_l(|y|)\}$ and a finite number of isolated positive eigenvalues. Assuming the existence of a local centre manifold, we look for a solution of (5.2) in the form

$$v(\tau) = a_l(\tau)\psi_l(y) + w(\tau) \quad \text{for } \tau \gg 1, \qquad w(\tau) = \mathrm{o}(a_l(\tau)) \in (E^c)^{\perp} \tag{5.11}$$

and obtain the asymptotic ODE

$$a_l' = -\delta_l |a_l|^{p-1} a_l (1 + \mathrm{o}(1)), \qquad \text{where } \delta_l = \langle \psi_l^p, \psi_l^* \rangle.$$

The positivity of the coefficient δ_l is of crucial importance (otherwise the behaviour is unstable) and is guaranteed at least for large l (see section 6). This gives

$$a_l(\tau) = \pm C_l \tau^{-1/(p-1)}(1 + \mathrm{o}(1)) \quad \text{as } \tau \to \infty, \qquad \text{where } C_l = \left(\frac{2m\delta_l}{l+N}\right)^{-(l+N)/2m}.$$

Returning to the original variables $\{x, t, u\}$ gives the following asymptotic patterns at $p = p_l$ (cf (1.11) for $l = 0$):

$$u(x,t) = \pm C_l (t \ln t)^{-(l+N)/2m}\left[\psi_l\left(\frac{|x|}{t^{1/2m}}\right) + \mathrm{o}(1)\right] \qquad \text{for } t \gg 1. \tag{5.12}$$

At the first critical exponent $p = p_0$, solutions (5.11) take the form

$$v(y,\tau) = \pm C_0 \tau^{-N/2m}[f(y) + \mathrm{o}(1)] \to 0 \qquad \text{as } \tau \to \infty.$$

Since this is the centre manifold behaviour (and $W^u(0) = \emptyset$), the trivial solution $v \equiv 0$ of (5.2) is stable, implying non-existence of a non-trivial generically stable VSS extending corollary 5.2 to $p = p_0$. The three types of asymptotic solutions described above are expected to form an 'evolutionary complete' countable subset of patterns in the diffusion–absorption problem under consideration. This kind of completeness is understood in a natural sense that any non-trivial X-valued solution takes, as $t \to \infty$, the form of one of such patterns. Evolutionary completeness remains a hard open problem for many second and all the higher-order semilinear parabolic equations posed as infinite-dimensional dynamical systems.

6. Very singular similarity profiles in the subcritical range $p \in (1, p_0)$

We return to the VSS profiles V satisfying the ODE problem (1.14), (1.15) in the subcritical range $p \in (1, p_0)$. Based on the linear analysis of sections 4 and 5, we consider bifurcation problems, in which we can construct solutions by parameter continuation and study their stability near critical values p_l.

6.1. *Bifurcations at $p = p_l$: local existence and stability of the VSS*

Taking p near the critical values as defined in (5.7), we look for small solutions of the problem (1.14), (1.15). At $p = p_l$, the linear operator $\mathbf{B}_1$ has a non-trivial kernel, hence, the following result.

Proposition 6.1. *Let for an integer $l \geqslant 0$, the eigenvalue $\lambda_l = -l/2m$ of operator (1.9) be of an odd multiplicity. Then, the critical exponent $p_l = 1 + (2m/(l+N))$ in (5.7) is a bifurcation point for the problem (1.14), (1.15).*

Proof. For a moment, given an $n \gg 1$, we denote by $(V^p)_n$ a suitable uniformly Lipschitz continuous truncation of the nonlinearity V^p such that $(V^p)_n \equiv V^p$ for $|V| \leqslant n$ so that

$$(V^p)_n \to V^p \qquad \text{as } n \to \infty \text{ uniformly on compact subsets.}$$

Consider in L^2_ρ the truncated equation

$$\hat{\mathbf{B}}V = -(1+c_1)V + (V^p)_n, \qquad \hat{\mathbf{B}} = \mathbf{B}_1 - (1+c_1)I \equiv \mathbf{B} - I. \tag{6.1}$$

The spectrum of $\hat{\mathbf{B}}$ is a translation of that of $\mathbf{B}$, (5.3), $\sigma(\hat{\mathbf{B}}) = \{-1 - (l/2m)\}$, and consists of strictly negative eigenvalues. The inverse operator $\hat{\mathbf{B}}^{-1}$ is known to be compact (proposition 2.4 in [13]). Therefore, in the corresponding integral equation

$$V = \mathbf{A}(V) \equiv -(1+c_1)\hat{\mathbf{B}}^{-1}V + \hat{\mathbf{B}}^{-1}(V^p)_n, \tag{6.2}$$

the right-hand side contains a compact Hammerstein operator, [32, chapter V] (see details in [3]). Bifurcations in the truncated problem (6.2) occur if the derivative $\mathbf{A}'(0) = -(1+c_1)\hat{\mathbf{B}}^{-1}$ has the eigenvalue 1 of odd multiplicity (see [32,33]). Since $\sigma(\mathbf{A}'(0)) = \{(1+c_1)/(1+(l/2m))\}$, we arrive at critical values (5.7). By construction, the solutions of (6.2) for $p \approx p_l$ are small in L^2_ρ and, as can be seen from the properties of the inverse operator, in H^{2m}_ρ. Since the weight (4.1) is a monotone growing function as $|y| \to \infty$, using the known asymptotic properties of solutions of the ODE (1.14) (section 3), $V \in H^{2m}_\rho$ is a uniformly bounded, continuous function. [It is worth mentioning that, for even m, solutions of (1.14) may blow-up at finite y (a striking contrast to second-order ODEs) forming singularities $\notin L^2_\rho$ locally.] Therefore, for $p \approx p_l$, we have bounded, small solutions only. Hence the same bifurcations occur in the original non-truncated equation (6.2) corresponding to $n = \infty$. □

Thus, $l = 0$ is always a bifurcation point since $\lambda_0 = 0$ is simple. In general, for $l = 1, 2, \ldots$, the odd multiplicity occurs depending on the dimension N. In particular, for $l = 1$, the multiplicity is N, and for $l = 2$, it is $(N(N+1))/2$. In the case of even multiplicity of λ_l, an extra analysis is necessary to guarantee that a bifurcation occurs, [33]. It is important that, for key applications, namely, for $N = 1$ and for the radial setting in $\mathbf{R}^N$, the eigenvalues (4.2) are simple and (5.7) are bifurcation points. Since the nonlinear perturbation term in the integral equation (6.2) is an odd sufficiently smooth operator, we arrive at the following result describing the local behaviour of bifurcation branches (see [32] and [33, chapter 8]).

Proposition 6.2. *Let λ_l be a simple eigenvalue of $\mathbf{B}$ with eigenfunction ψ_l. Denoting*

$$\kappa_l = \langle \psi_l^p, \psi_l^* \rangle, \tag{6.3}$$

we have that problem (1.14), (1.15) has (i) precisely two small solutions for $p \approx p_l^-$ and no solutions for $p \approx p_l^+$ if $\kappa_l > 0$, and (ii) precisely two small solutions for $p \approx p_l^+$ and no solutions for $p \approx p_l^-$ if $\kappa_l < 0$.

In order to describe the asymptotics of solutions as $p \to p_l$, we apply the Lyapunov–Schmidt method ([33, chapter 8]) to equation (6.2) with the operator $\mathbf{A}$ being differentiable at 0. Since, under the assumptions of proposition 6.2, the kernel $E_0 = \ker \mathbf{A}'(0) = \operatorname{Span}\{\psi_l\}$ is one dimensional, denoting by E_1 the complementary (orthogonal to ψ_l^*) invariant subspace, we set $V = V_0 + V_1$, where $V_0 = \varepsilon_l \psi_l \in E_0$ and $V_1 = \sum_{k \neq l} \varepsilon_k \psi_k \in E_1$. Let P_0 and P_1, $P_0 + P_1 = I$, be projections onto E_0 and E_1, respectively. Projecting (6.2) with $n = \infty$ onto E_0 yields

$$\gamma_l \varepsilon_l = \langle \hat{\mathbf{B}}^{-1}(V^p), \psi_l^* \rangle, \qquad \gamma_l = 1 - \frac{1+c_1}{1+(l/2m)} = -\frac{(N+l)s}{(p-1)(2m+l)}, \tag{6.4}$$

where $s = p_l - p$. By the general bifurcation theory (see, e.g. [33, p 355] and [11, p 383], note that operator $\mathbf{A}'(0)$ is Fredholm of index zero), the equation for V_1 can be solved and this gives

$V_1 = \mathrm{o}(\varepsilon_l)$ as $\varepsilon_l \to 0$, so that ε_l is calculated from the Lyapunov bifurcation equation (6.4) as follows:

$$\gamma_l \varepsilon_l = \varepsilon_l^p \langle \hat{\mathbf{B}}^{-1} \psi_l^p, \psi_l^* \rangle + \mathrm{o}(\varepsilon_l^p) \implies |\varepsilon_l|^{p-1} = \hat{c}_l[(p_l - p) + \mathrm{o}(1)], \qquad \hat{c}_l = \frac{(l+N)^2}{4m^2\kappa_l},$$

where we have performed calculations as follows:

$$\langle \hat{\mathbf{B}}^{-1} \psi_l^p, \psi_l^* \rangle = \langle \psi_l^p, (\hat{\mathbf{B}}^*)^{-1} \psi_l^* \rangle = -\frac{\kappa_l}{1 + l/2m}.$$

It is natural to require $\kappa_l > 0$. In view of the orthonormality property (4.6), for $p = 1$ we have $\kappa_l = 1$, so that by continuity we can guarantee that

$$\kappa_l > 0 \qquad \text{at least for all } p \approx 1^+. \tag{6.5}$$

Thus, we obtain a countable sequence of bifurcation points (5.7) satisfying $p_l \to 1^+$ as $l \to \infty$, with typical pitch-fork bifurcation branches appearing in a left-hand neighbourhood, for $p < p_l$. The asymptotic behaviour of solutions takes the form

$$V_l(y) = \pm[\hat{c}_l(p_l - p)]^{1/(p-1)}(\psi_l(y) + \mathrm{o}(1)) \qquad \text{as } p \to p_l^-. \tag{6.6}$$

We now prove the main result concerning 'local' existence and stability of the VSS solution with the similarity profile $V_0(y)$ corresponding to the first bifurcation point, $p = p_0$. If $\kappa_0 > 0$, as expected, then two bifurcation branches exist for $p < p_0$.

Theorem 6.3. *For $p \approx p_0^-$, problem (1.14), (1.15) admits a solution $V_0(y)$ provided that $2m/N$ is small enough, and then it is an asymptotically stable stationary solution of the rescaled equation (5.2).*

Proof. As we have shown, a continuous branch bifurcating at $p = p_0^-$ exists if

$$\kappa_0 = \langle \psi_0^{p_0}, \psi_0^* \rangle \equiv \int |f|^{2m/N} f > 0 \qquad (\psi_0^* \equiv 1). \tag{6.7}$$

In view of the positivity dominance of the rescaled fundamental solution f, $\int f = 1$, we have that (6.7) holds by continuity provided that $2m/N \ll 1$. Therefore, in this case there exists a solution (6.6) with $l = 0$ satisfying for small $s = p_0 - p > 0$ uniformly

$$V_0(y) = (\hat{c}_0 s)^{1/(p-1)}[f(y) + \mathrm{o}(1)], \qquad \hat{c}_0 = \frac{N^2}{4m^2\kappa_0}. \tag{6.8}$$

We now estimate the spectrum of the linearized operator of equation (5.2):

$$\mathbf{D}_0 = \mathbf{B}_1 - p|V_0|^{p-1} I. \tag{6.9}$$

Some of the eigenvalues of (6.9) follow from the original PDE (1.1). For instance, the stable eigenspace with $\hat{\lambda} = -1$, $\hat{\psi} = (1/(p-1))V_0 + (1/2m)\nabla V_0 \cdot y \in L^2_\rho$, follows from the time-translational invariance of the PDE. For $N = 1$, translations in x yield another pair $\hat{\lambda} = -(1/2m)$, $\hat{\psi} = V_{0y} \in L^2_\rho$. For $N > 1$, in the non-radial setting, this $\hat{\lambda}$ has multiplicity N with eigenfunctions V_{0y_i}. These are not the first pair with the maximal real part. Bearing in mind that the spectrum of the unperturbed operator $\mathbf{B}$ is real, (4.2), and has the unique, non-hyperbolic eigenvalue $\lambda_0 = 0$, we use (6.8) to obtain

$$\mathbf{D}_0 = \mathbf{B} + s(1 + \mathrm{o}(1))\mathbf{C}, \tag{6.10}$$

where, as it follows from (6.7) and (6.8) at $p = p_0$, the perturbation has the form

$$\mathbf{C} = \frac{N^2}{4m^2}\left(1 - \frac{p_0}{\kappa_0}|f|^{2m/N}\right) I. \tag{6.11}$$

Therefore, we consider the spectrum of the perturbed operator

$$\tilde{\mathbf{D}}_0 = \mathbf{B} + s\mathbf{C}. \tag{6.12}$$

Since $(\mathbf{B} - I)^{-1}\mathbf{C}$ is bounded,

$$(\tilde{\mathbf{D}}_0 - I)^{-1} = (I + s(\mathbf{B} - I)^{-1}\mathbf{C})^{-1}(\mathbf{B} - I)^{-1},$$

is compact for small $|s|$ as the product of compact and bounded operators. Hence, $\tilde{\mathbf{D}}_0$ also has only a discrete spectrum. By the classical perturbation theory of linear operators (see, e.g., [26]), the eigenvalues and eigenvectors of $\tilde{\mathbf{D}}_0$ can be constructed as a perturbation of the discrete spectrum $\sigma(\mathbf{B})$ consisting of eigenvalues of finite multiplicity. We are interested in the perturbation of the first simple eigenvalue $\lambda_0 = 0$. Setting

$$\tilde{\lambda}_0 = s\mu_0 + \mathrm{o}(s), \qquad \tilde{\psi}_0 = \psi_0 + s\varphi_0 + \mathrm{o}(s) \qquad \text{as } s \to 0$$

and substituting these expansions in the eigenvalue equation $\tilde{\mathbf{D}}_0\tilde{\psi}_0 = \tilde{\lambda}_0\tilde{\psi}_0$ yields

$$\mathbf{B}\varphi_0 = (-\mathbf{C} + \mu_0 I)\psi_0. \tag{6.13}$$

We then obtain the solvability (orthogonality) condition

$$\langle(-\mathbf{C} + \mu_0 I)\psi_0, \psi_0^*\rangle = 0 \implies \mu_0 = \langle\mathbf{C}f, 1\rangle.$$

Using (6.11) yields $\mu_0 = -N/2m < 0$. Therefore, $\mathrm{Re}\,\tilde{\lambda}_0 < -Ns/4m < 0$ for all $p \approx p_0^-$. Since, with these properties of the spectrum, the perturbation (6.9) of $\mathbf{B}$ remains a sectorial operator with $\sigma(\tilde{\mathbf{D}}_0) \subset \{\mathrm{Re}\,\lambda \leqslant -Ns/4m\}$ and $\|\mathrm{e}^{\tilde{\mathbf{B}}_1\tau}\|_{\mathcal{L}} \leqslant C\,\mathrm{e}^{-N(p_0-p)\tau/4m}$ in the norm of $\mathcal{L}(H^{2m}_\rho, H^{2m}_\rho)$ [20], $V_0(y)$ is exponentially stable in H^{2m}_ρ. □

We expect that condition (6.7) remains valid for any m and N so that $V_0(y)$ is stable without the restriction $2m \ll N$. We have strong numerical support for this, but, as yet, no rigorous proof exists. Possibly, to check conditions such as (6.7) we must currently rely on numerical evidence and then, as often happens in spectral theory and applications, theorem 6.3 can be established with a hybrid analytic-computational proof. We also expect that the whole branch bifurcating from $p = p_0$ remains stable for all $p \in (1, p_0)$, though the proof would require us to establish that the discrete spectrum $\sigma(\mathbf{D}_0)$ never touches the imaginary axis. In particular, this difficult open problem means that a new (nonlinear) saddle-node bifurcation never occurs on this p_0-branch, i.e. it does not have turning points. This is valid [41] for the variational problem with $m = 1$, as well as for ordinary differential higher-order equations with self-adjoint positive operators of special structure of quasi-derivatives [39]. Further, one can see that the other bifurcation branches are *unstable*. Taking any $l \geqslant 1$, instead of (6.10) we now have

$$\mathbf{D}_l = \mathbf{B}_1 - p|V_l|^{p-1}I \equiv \mathbf{B} + [c_1 - sp_l\hat{c}_l(|\psi_l|^{p-1} + \mathrm{o}(1))]I, \qquad s = p_l - p.$$

From the definition of $\mathbf{B}_1$, (1.16), $c_1 > 0$ for all $p \approx p_l$; thus, V_l for $l \geqslant 1$ is unstable.

The transition to a subset of linear patterns at $p = 1$. By (5.7), the sequence of critical exponents converges from above to $p = 1$ corresponding to the linear equation

$$\bar{u}_t = -(-\Delta)^m\bar{u} - \bar{u}. \tag{6.14}$$

Therefore, expansion (6.6) gives an exceptional possibility to see the *transition* from the nonlinear patterns for $p > 1$ to the linear ones for $p = 1$. For $p \approx 1^+$ there exists an arbitrarily large number of nonlinear patterns which become unbounded linear patterns in the case $p = 1$ (this trend is seen in figure 1 to be discussed below). Some simple computations reveal the scaling factors of such a transition. It is easy to get all the linear (radial) patterns. Setting $\bar{u} = \mathrm{e}^{-t}\tilde{u}$ in (6.14) yields the linear parabolic equation (1.8) with the known countable subset

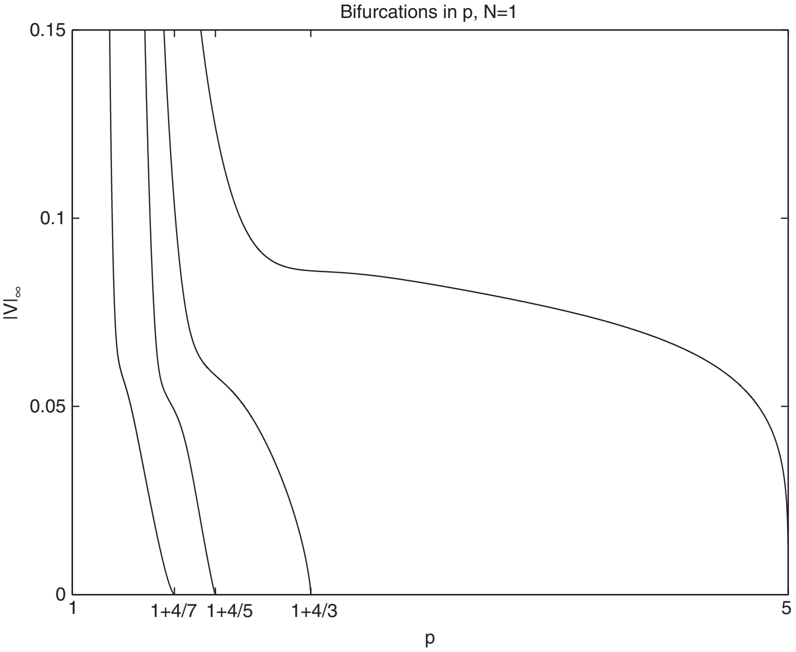

Figure 1. Bifurcation diagram with respect to p, $m = 2$, $N = 1$.

of asymptotic patterns (cf (5.10)). Hence, up to a constant multiplier, the subset of linear patterns is

$$\bar{u}_l(x,t) = \mathrm{e}^{-t} t^{-(N+l)/2m} \psi_l(y), \qquad y = \frac{x}{t^{1/2m}}; \quad l = 0, 1, 2, \ldots . \tag{6.15}$$

It is important that due to the completeness of the eigenfunctions, lemma 4.1, the pattern subset (6.15) is evolutionary complete (these notions coincide in the linear case). On the other hand, using expansion (6.6), we obtain the following nonlinear asymptotic patterns corresponding to the original PDE (1.1) for $p \approx 1^+$:

$$u_l(x,t) \approx t^{-1/(p-1)} [\hat{c}_l (p_l - p)]^{1/(p-1)} \psi_l(y). \tag{6.16}$$

Then, the scaling factors in the pattern transition as $p \to 1^+$ are given by

$$[t^{-1} \hat{c}_l (p_l - p)]^{-1/(p-1)} u_l(x,t) \to \psi_l(y) \equiv \mathrm{e}^t t^{(N+l)/2m} \bar{u}_l(x,t).$$

For the ODE (1.14), the nonlinear VSS profiles are generated at $p = 1^+$ by the eigenfunctions of a linear Sturm–Liouville problem for the non-self-adjoint operator $\mathbf{B}$ (cf (6.6)).

6.2. On the global bifurcation p-diagram

Our counting argument on the number of available solutions for a particular value of p is based on the number of bifurcation points (the Morse index of $\mathbf{B}_1$) which are available for $p < p_0$. Because of the existence of infinitely many solutions in the linear problem for $p = 1$, we conjecture that each branch bifurcating from $1 < p_l \leqslant p_0$ is global in p and contributes one additional exponentially decaying solution. Numerical results presented in figure 1 support this as the branches are monotonic in p with no branch ever contributing more than one solution. It is interesting that the asymptotic expansion (6.6) for $p \approx p_l^-$ containing the function

$$h_l(p) = (p_l - p)^{1/(p-1)}, \tag{6.17}$$

approximately describes the singular behaviour of the bifurcation branches as $p \to 1^+$ provided that $p_l > 1$. More precisely, equation (1.14) shows that as $p \to 1^+$ the singularity is generated by the term containing $1/(p-1)$. Introducing the small parameter $\varepsilon = p - 1 > 0$ and using the scaling

$$V(y) = \varepsilon^{-1/\varepsilon} W_\varepsilon(y), \tag{6.18}$$

one obtains the equation

$$\varepsilon \left[-(-\Delta)^m W_\varepsilon + \frac{1}{2m} \nabla W_\varepsilon \cdot y \right] + W_\varepsilon - |W_\varepsilon|^\varepsilon W_\varepsilon = 0.$$

Since $W_\varepsilon - |W_\varepsilon|^\varepsilon W_\varepsilon = -\varepsilon W_\varepsilon \ln |W_\varepsilon| + \mathrm{O}(\varepsilon^2)$ uniformly on any subset where W_ε is bounded and bounded away from zero, we obtain in the limit $\varepsilon \to 0^+$ that W_0 (if it exists) solves the equation

$$-(-\Delta)^m W_0 + \frac{1}{2m} \nabla W_0 \cdot y - W_0 \ln |W_0| = 0. \tag{6.19}$$

We expect that scaling (6.18) (cf (6.18)) correctly describes the leading factor of the exponential blow-up of bifurcation branches as $p \to 1^+$. Equation (6.19) is not essentially simpler than the original one (1.14) and its solvability remains an open problem for $m \geqslant 2$. Similar global properties of bifurcation branches have been rigorously established for variational semilinear second-order elliptic problems; cf [41, theorem 1.2] and the corresponding bifurcation diagram. For higher-order equations, similar results are known for special problems with positive self-adjoint operators [39]. These results do not apply here though the bifurcation diagram looks similar. To compute numerical approximations, we begin with numerical approximations to (6.6) near the critical exponents p_l. These are then continued in p using the pseudo-arclength package AUTO [12]. All the numerically constructed similarity profiles $V(y)$ have asymptotics from the exponential bundle described in section 3. This was not enforced numerically but is simply a property of the initial profiles used for continuation. Such solutions correspond to sharp agreement in the number of available solutions described in the estimate (1.17).

6.3. *VSS similarity profiles for various p, N and m*

The two profiles with $m = 2$, $p = 2$ and $N = 1$ are presented in figure 2(*a*). Qualitatively, the picture remains unchanged in higher dimensions. In figure 2(*b*) we present radially symmetric profiles $V(|y|)$ for $N = 2$ and 3. Topologically, there is no great distinction between the computed solutions for different values of N with all branches emanating from the bifurcation points of the associated linear operator. Further, the solutions and bifurcation diagrams are qualitatively similar for $m = 3$.

6.4. *On μ-bifurcations and the local bifurcation diagram*

As another branching approach, we introduce the linear operator

$$\mathcal{L}_\mu = -(-\Delta)^m + \mu y \cdot \nabla + \frac{1}{p-1} I \qquad \text{with a parameter } \mu \geqslant 0 \tag{6.20}$$

and consider the corresponding equation

$$\mathcal{L}_\mu V - V^p = 0 \qquad \text{in } \mathbf{R}^N \text{ with condition (1.15).} \tag{6.21}$$

For $\mu = 1/2m$ we get the original VSS problem (1.14), (1.15). This μ-parameterization provides an additional approach to study the multiplicity of the VSS profiles. It is important for describing a transition to the elliptic problem occurring at $\mu = 0$:

$$-(-\Delta)^m V + \frac{1}{p-1} V - V^p = 0 \qquad \text{in } \mathbf{R}^N, \tag{6.22}$$

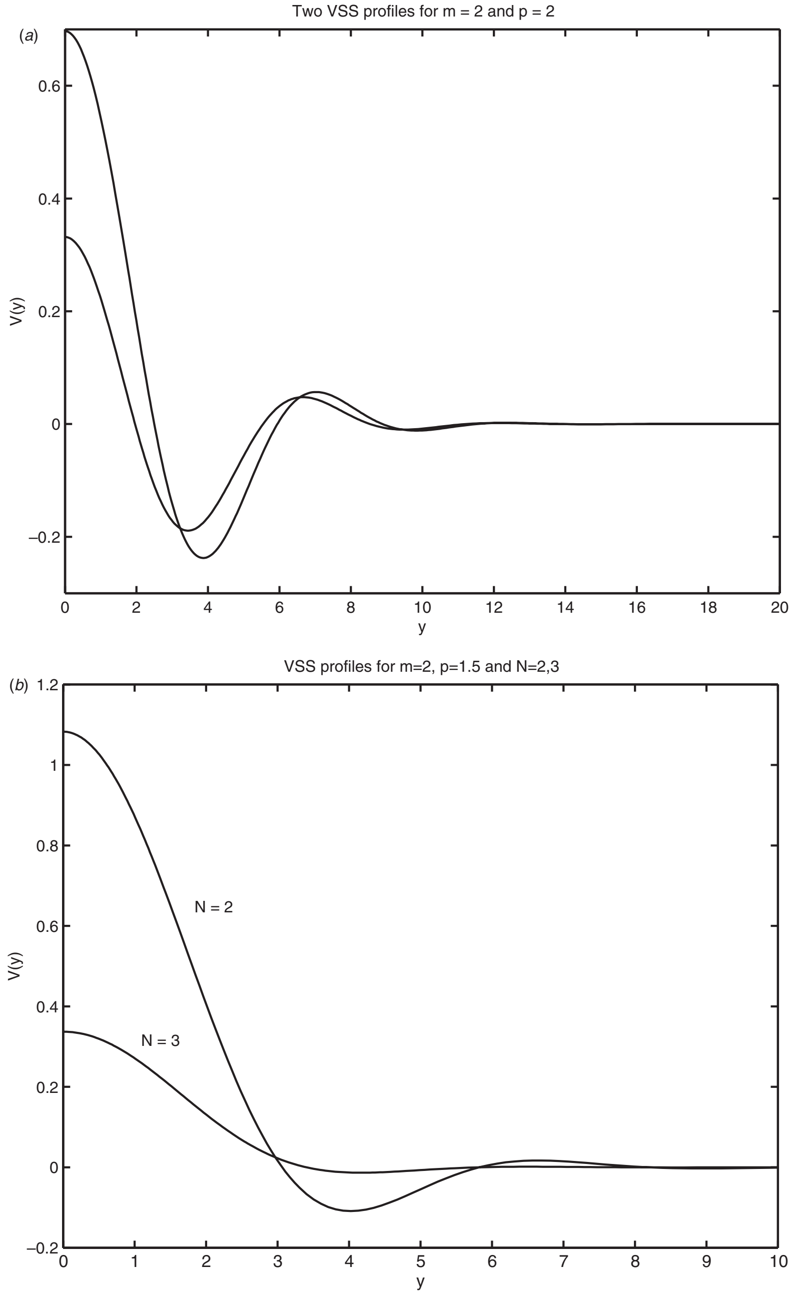

Figure 2. (*a*) Two similarity profiles V_0 and V_2 for $m = 2$, $p = 2$, $N = 1$. (*b*) The primary similarity profile V_0 for $m = 2$, $p = \frac{3}{2}$, $N = 2$ and 3.

which admits a variational formulation in terms of critical points of the functional $\Phi(V) = -\frac{1}{2}\|\bar{D}^m V\|_2^2 + (1/(2(p-1)))\|V\|_2^2 - (1/(p+1))\|V\|_{p+1}^{p+1}$. For $N = 1$, (6.22) is a Hamiltonian dynamical system with known solution properties for $m = 2$ (see [38] and references therein). In [3], the μ-bifurcation diagrams were shown to be quite effective in studying blow-up similarity profiles for the semilinear $2m$th-order ODE occurring in the PDE (1.1) with the source term $+u^p$. This changes the sign in the last three terms in the ODE (1.14) leading to completely different properties of the asymptotic bundle and similarity profiles V, and quite a rich structure of the Hamiltonian system at $\mu = 0$, where infinitely many solutions are known to exist. We now briefly describe bifurcation in the problem (6.21) from the trivial solution $V^0 \equiv 0$. We find the spectrum of $\mathcal{L}_\mu$ in L^2_ρ by introducing the new independent variable $y = z/(2m\mu)^{1/2m}$. Then, lemma 4.1 yields

$$\mathcal{L}_\mu = 2m\mu\mathbf{B} + \left(-\mu N + \frac{1}{p-1}\right) I \implies \sigma(\mathcal{L}_\mu) = \left\{-\mu(N+l) + \frac{1}{p-1}\right\},$$

whence the following result, analogous to proposition 6.2.

Proposition 6.4. *Let the eigenvalue* $\lambda_l = -l/2m$ *in (4.2) be of odd multiplicity. Then,*

$$\mu_l = \frac{1}{(N+l)(p-1)} \tag{6.23}$$

is a bifurcation point in the problem (6.21).

Indeed, denoting $c = 1/(p-1) - \mu N$ and $\hat{\mathcal{L}}_\mu = \mathcal{L}_\mu - (1+c)I$, we have from (6.21)

$$\hat{\mathcal{L}}_\mu V = -(1+c)V + V^p.$$

The spectrum $\sigma(\hat{\mathcal{L}}_\mu) = \{-1 - \mu l\}$ consists of strictly negative eigenvalues. The proof is now analogous to the integral equation

$$V = \mathbf{A}_\mu(V) \equiv -(1+c)\hat{\mathcal{L}}_\mu^{-1} V + \hat{\mathcal{L}}_\mu^{-1}(V^p). \tag{6.24}$$

Bifurcations occur if the derivative $\mathbf{A}'_\mu(0) = -(1+c)\hat{\mathcal{L}}_\mu^{-1}$ has the eigenvalue 1 of odd multiplicity. Since $\sigma(\mathbf{A}_\mu{}'(0)) = \{(1+c)/(1+\mu l)\}$, this yields (6.23). Again, by construction, the solutions of (6.24) for $\mu \approx \mu_l$ are small in L^2_ρ, H^{2m}_ρ and uniformly. The local bifurcation structure in μ is similar to that for p (cf proposition 6.2). Namely, if λ_l is a simple eigenvalue of $\mathbf{B}$, then (6.2) has (i) precisely two small solutions for $\mu \approx \mu^-$ and no solutions for $\mu \approx \mu^+$ if $\kappa_l > 0$ and (ii) precisely two small solutions for $\mu \approx \mu^+$ and no solutions for $\mu \approx \mu^-$ if $\kappa_l < 0$. Under the assumption $\kappa_l > 0$, pitch-fork bifurcations occur with branches appearing in a left-hand neighbourhood, for $\mu < \mu_l$, with the behaviour

$$V(y) = \pm[c_l(\mu_l - \mu)]^{1/(p-1)}[\psi_l(y) + o(1)], \qquad \mu \to \mu_l^-; \quad c_l = \frac{N+l}{\kappa_l}. \tag{6.25}$$

We apply numerical methods to extend the local behaviour of the bifurcation diagram from the previous section. We will fix $N = 1$ and $m = 2$, and using initial data near the bifurcation points, continue solutions in μ using the numerical continuation code AUTO [12]. Taking $p = 2$, from (1.17) we expect at least two solutions. We conjecture that these solutions arise from the two unstable modes of the operator $\mathbf{B}_1$ in this case. Notice in figure 3 that the two solutions available at $\mu = \frac{1}{4}$ extend from branches connecting bifurcation points on either side of the value $\mu = \frac{1}{4}$, $1 \to \frac{1}{5}$ and $\frac{1}{3} \to \frac{1}{7}$. There are no other bifurcation points available greater than $\frac{1}{4}$. All branches presented in figure 3 leave the bifurcation points (6.23) as predicted by the asymptotics (6.25). However, for $\mu = 0$, as a generic property, the Hamiltonian systems like (6.22) do not admit non-trivial solutions as homoclinics of zero (see [38, p 196] and references therein). This gives rise to the closed orbits observed numerically in figure 3.

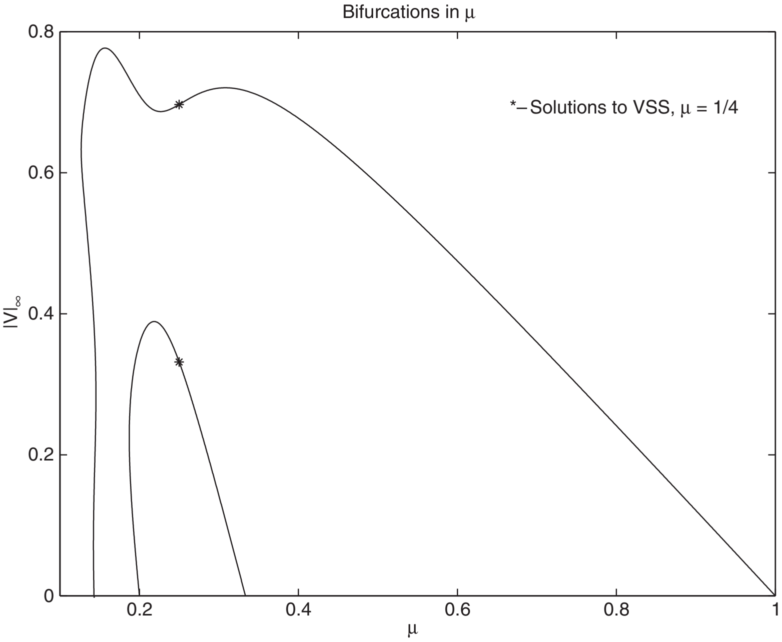

Figure 3. Bifurcation diagram with respect to μ.

7. For $m > 1$ the rescaled operator is not potential in weighted spaces

Let us return to the general problem of stability of similarity profiles constructed above locally close to bifurcation points. Assuming that $1 < p < p_0$, we consider the rescaled PDE (5.2) having VSS profiles $\{V_l\}$ as the stationary solutions. We have conjectured that the first VSS similarity profile $V_0(y)$, corresponding to the branch bifurcating from $p = p_0^-$, is the only generically stable one describing the asymptotic behaviour for a dense subset of initial data $v_0 \in H_\rho^{2m} \cap L^\infty$. In other words, the stable branch, which was proved in theorem 6.3 to originate at $p = p_0^-$, remains stable for all $1 < p < p_0$. We have strong numerical evidence supporting this conclusion. For $m = 1$, this is proved by various methods; see first results by the Lyapunov method (parabolic monotonicity) in [23] and the references in [40, chapter 2]. Even for $N = 1$ we do not know if equation (5.2) admits a Lyapunov function and is a gradient system. Moreover, we will present evidence to suggest that this is not the case for any $m > 1$.

Gradient systems for $m = 1$. This is in striking contrast to the case of second-order equations, where Lyapunov functions often play a key role in the asymptotic analysis of blow-up and global solutions of quasilinear heat equations. For the rescaled equations such as (5.2) with $m = 1$ written in symmetric form

$$v_\tau = \frac{1}{\rho}\nabla \cdot (\rho \nabla v) + \frac{1}{p-1} v - v^p, \qquad \rho = \mathrm{e}^{|y|^2/4}, \tag{7.1}$$

the potential (variational) structure of the operator in L_ρ^2 was used in [17] and in a number of subsequent papers; see references in [7, 8], [40, chapter 2]. Indeed, since the linear operator in (7.1) is self-adjoint in L_ρ^2 (it is a classical operator of mathematical physics, see [4], p 48), a Lyapunov function is obtained by multiplying by ρv_τ and integrating over $\mathbf{R}^N$. Moreover, for the second-order parabolic equations with one spatial variable (or in $\mathbf{R}^N$ with radial symmetry), potential operators are dominant. Recall [43] that any quasilinear second-order uniformly parabolic equation with smooth coefficients

$$v_\tau = \mathbf{P}_2(v) \equiv a(y, v, v_y)v_{yy} + b(y, v, v_y) \tag{7.2}$$

on a bounded interval with typical (nonlinear) boundary conditions is a gradient system. Namely, there exists a smooth multiplier $\rho(y, v, w) > 0$, $w = v_y$, such that $\rho\mathbf{P}_2(v) = F'(v)$ is a potential operator. Hence, multiplying (7.2) by ρv_τ and integrating in y yields a Lyapunov function, and for some function $\Phi(y, v, v_y)$ ($F(v) = \int \Phi(v)$ is the potential of $\rho\mathbf{P}_2(v)$), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int \Phi = \int \rho v_\tau^2 \geqslant 0 \tag{7.3}$$

on bounded evolution orbits. Indeed, using formal integration by parts, we get

$$\begin{aligned}\int \rho(av_{yy} + b)v_\tau &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int \Phi(y, v, v_y) = \int(\Phi_v v_\tau + \Phi_w v_{y\tau})\\ &= \int(\Phi_v - D_y\Phi_w)v_\tau \equiv \int(\Phi_v - \Phi_{wy} - \Phi_{wv}v_y - \Phi_{ww}v_{yy})v_\tau,\end{aligned} \tag{7.4}$$

where $D_y = \mathrm{d}/\mathrm{d}y$ is the full derivative. Then, ρ and Φ satisfy the system of PDEs

$$\rho a = -\Phi_{ww}, \qquad \rho b = \Phi_v - \Phi_{wy} - \Phi_{wv}w.$$

Differentiating the second equation in w yields a linear first-order PDE for ρ

$$(\rho a)_y + (\rho a)_v w - (\rho b)_w = 0. \tag{7.5}$$

Introducing the new dependent variable $\rho = \mathrm{e}^P > 0$ leads to an inhomogeneous first-order PDE for P and the existence of ρ is proved by the standard method of characteristics.

Higher-order rescaled equations do not possess a natural gradient structure. Without loss of generality, we prove this negative result for equation (5.2) with $m = 2$ and $N = 1$ denoting $\gamma_1 = 1/2m$, $\gamma_2 = \gamma_1 + c_1$.

Proposition 7.1. *The fourth-order parabolic equation*

$$v_\tau = \mathbf{P}_4(v) = -v_{yyyy} + \gamma_1 y v_y + \gamma_2 v - v^p \equiv -v_{yyyy} + b(y, v, v_y, v_{yy}, v_{yyy}) \tag{7.6}$$

is not a gradient system in the sense that (7.3) *does not hold for any* $\Phi(y, v, v_y, v_{yy}) \equiv \Phi(y, v, w, z)$ *and* $\rho(y, v, v_y, v_{yy}, v_{yyy})$.

Proof. Performing differentiating formal integration by parts in (7.3), we obtain

$$\int \rho(-v_{yyyy} + b)v_\tau = \int(\Phi_v - D_y\Phi_w + (D_y)^2\Phi_z)v_\tau \equiv \int(\mathcal{L}_3\Phi + \Phi_{zz}v_{yyyy})v_\tau,$$

where $\mathcal{L}_3\Phi$ contains derivatives of v up to the third order, v_{yyy}. Comparing the terms with higher-order derivatives, this gives a system of PDEs

$$\rho = -\Phi_{zz}, \qquad \rho b = \mathcal{L}_3\Phi. \tag{7.7}$$

The first equation shows that $\rho = \rho(y, v, w, z)$ and then taking from the second equation the coefficient of the third-order derivative $v_{yyy} = r$ yields the following linear first-order PDE:

$$\rho_y + \rho_v w + \rho_w z + \tfrac{1}{2}\rho_z r = 0.$$

Hence, $\rho_z \equiv 0$, i.e. $\rho = \rho(y, v, w)$. Next, we get $\rho_w \equiv 0$, etc and finally we have $\rho_y = 0$, i.e. $\rho \equiv 1$, which does not satisfy equation (7.6) unless $\gamma_1 = 0$. This implies that the system (7.7) has no solution. □

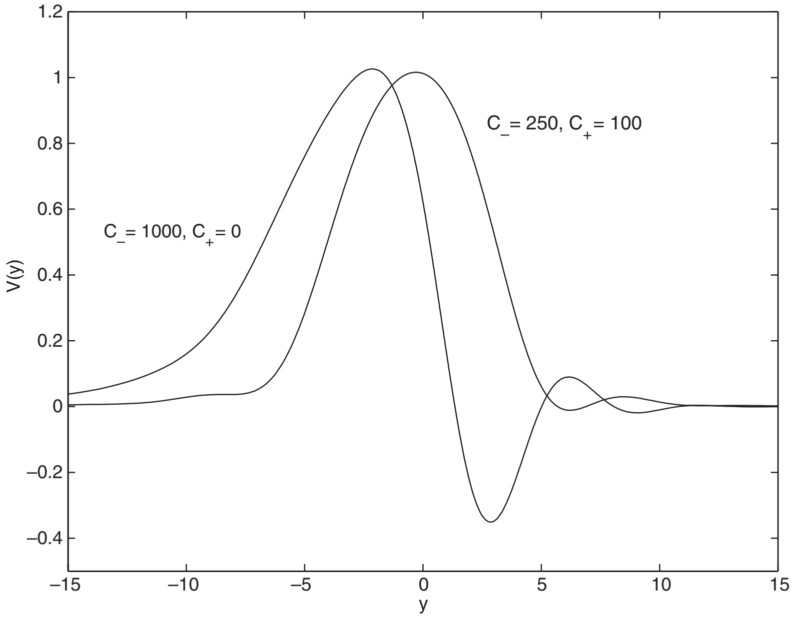

Figure 4. Examples of asymmetric solutions with algebraic decay; $m = 2$, $p = 2$ and $N = 1$.

Let us present another interpretation and extension of this negative result for any $m \geqslant 2$. In a given Hilbert space, a C^1 operator is potential iff its Frechet derivative is self-adjoint [33]. In particular, this implies that $\mathbf{P}'_{2m}(0)$ in (5.2) is self-adjoint in the given topology. Actually, this assumes that the linear operator $\mathbf{B}$ in (1.9) must admit a self-adjoint extension. It is not self-adjoint in any natural topology and admits a formal self-adjoint representation in a 'little' Hilbert space l^2_ρ of sequences $\{c_\beta\}$ representing the functions $v = \sum c_\beta \psi_\beta$, $c_\beta = \langle v, \psi^*_\beta \rangle$ with $\sum |c_\beta|^2 < \infty$ and with the corresponding scalar product in l^2 (see details in [13, section 3]). But for this case, the nonlinear term v^p in (5.2) or (7.6) is not potential in l^2_ρ. Thus, it seems that the only way to establish stability of the first VSS is to use the linearized operator which requires an estimate of the spectrum of the corresponding $2m$th-order non-self-adjoint operators with non-constant coefficients. We have proved this in theorem 6.3 for all $p \approx p_0$, provided that $2m/N \ll 1$. For arbitrary $p \in (1, p_0)$, this is an open problem, where sharp numerical methods can play an important role.

8. On continuous subsets of algebraically decaying similarity profiles

By the asymptotic analysis in section 3, the similarity ODE (1.14) admits solutions with algebraic decay (3.7) accompanying the multi-parametric exponential bundle of dimension m. Such similarity solutions (1.13) form a continuous subset of asymptotic patterns. Note that $V \in L^1$ if $p < p_0$ and $V \notin L^1$ for $p \geqslant p_0$. Obviously, $V \notin L^2_\rho$ so that such similarity solutions have different domains of stability from the VSSs. In particular, the following simple but, in our opinion, typical characterization of their domain of stability holds.

Proposition 8.1. *Let $p > p_* = 1+(4m/N)$ and let V be a similarity profile in (1.13) satisfying (1.14), (3.7) for some constant $C \neq 0$. Let $u(x, t)$ be a solution of equation (1.1) such that $u(\cdot, t) - u_*(\cdot, 1 + t) \in H^{2m}$ for all $t \geqslant 0$. Then,*

$$\theta(y, t) \equiv (1 + t)^{1/(p-1)} u(y(1 + t)^{1/2m}, t) \to V(y) \qquad \text{in } L^2 \text{ as } t \to \infty. \tag{8.1}$$

Proof. Multiplying the equation for difference $w = u - u_*$, $w_t = -(-\Delta)^m w - (u^p - u_*^p)$, by w and integrating over $\mathbf{R}^N$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_2^2 = -\int |\bar{D}^m w|^2 - \langle u^p - u_*^p, u - u_*\rangle \leqslant 0,$$

so that $\|w(t)\|_2^2 \leqslant C$ for $t > 0$. By scaling, $\|u - u_*\|_2^2 \equiv (1+t)^\nu \|\theta - V\|_2^2$, with exponent $\nu = -2/(p-1) + N/2m > 0$, and hence (8.1) follows. □

In figure 4 we present an example of profiles from this continuous family, which is extremely wide. Namely, for $N = 1$, we expect that there are non-symmetric profiles $V(y)$ satisfying

$$|y|^{2m/(p-1)} V(y) \to C_\pm \qquad \text{as } y \to \pm\infty, \quad \text{with arbitrary } C_\pm \in \mathbf{R}. \tag{8.2}$$

In $\mathbf{R}^N$, the variety of such solutions is measured by an arbitrary smooth function $\chi \not\equiv 0$ on the unit sphere S_1, and we expect that there exists a solution (possibly unique) of the elliptic equation (1.14) satisfying

$$V(y) = \left[\chi\left(\frac{y}{|y|}\right) + \mathrm{o}(1)\right] |y|^{-2m/(p-1)} \qquad \text{as } y \to \infty. \tag{8.3}$$

$\chi \equiv 0$ corresponds to VSS profiles with exponential decay. For $m = 1$, in some parameter ranges, such a result is proved by a PDE analysis using the fact that such similarity solutions are generated by initial data $u_0(x) = |x|^{-2m/(p-1)}\chi(x/|x|)$ and the solvability and uniqueness of solutions is established by using the Maximum principle, and comparison with super- and sub-solutions play a key role (see [30]).

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