A proof of a partition conjecture of Bateman and Erdős

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Proposed Running Head: Bateman-Erdős conjecture

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Abstract

Bateman and Erdős found necessary and sufficient conditions on a set $A$ for the $k$’th differences of the partitions of $n$ with parts in $A$, $p_A^{(k)}(n)$, to eventually be positive; moreover, they showed that when these conditions occur $p_A^{(k+1)}(n)/p_A^{(k)}(n)$ tends to zero as $n$ tends to infinity. Bateman and Erdős conjectured that the ratio $p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(n^{-1/2})$. We prove this conjecture.

Key words: Asymptotic Enumeration, partitions.

1 Introduction

Let $A$ be a non-empty set of positive integers. Throughout this paper, $p_A(n)$ will denote the number of partitions of $n$ with parts from $A$ and $p_A^{(k)}(n)$ will denote the $k$’th difference of $p_A(n)$. That is to say

$$p_A^{(k)}(n) = [x^n](1 - x)^k \prod_{a \in A} (1 - x^a)^{-1}.$$ 

We shall say, as do Bateman and Erdős, that a subset $A$ of the natural numbers has property $P_k$, if there are more than $k$ elements in $A$, and if any subset of $k$ elements is removed from $A$, the remaining elements have gcd one. Bateman and Erdős say that if $k < 0$, then any non-empty set of positive numbers has property $P_k$. Bateman and Erdős [1] showed that $p_A^{(k)}(n)$ is eventually positive if and only if $A$ has property $P_k$. At the end of their paper, Bateman and Erdős made a conjecture about the behavior of the $k$’th differences of $p_A(n)$. We now state this conjecture.
**Conjecture 1** (Bateman-Erdős) If a set $A$ of positive integers has property $P_k$, then $p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(n^{-1/2})$.

Bateman and Erdős prove (see Theorem 3 of [1]) the conjecture when $A$ is a finite set; in fact, when $A$ is a finite set with property $P_k$ they show that

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) = O(1/n).$$

We shall show their conjecture is true when $A$ is infinite. The best known conditions under which the Bateman-Erdős conjecture has been verified are due to Richmond [3], who, using the saddle point method, obtained asymptotics of certain partition functions that allowed him to prove the Bateman-Erdős conjecture for certain sets $A$. Bateman and Erdős observed (see page 12 of [1]) that if the conjecture were true, then it would be the best possible result. To see this, take $A = \{1, 2, 3, \ldots\}$. By Rademacher’s exact formula for $p_A(n)$ (see [2]) it follows that

$$p_A^{(k+1)}(n)/p_A^{(k)}(n) = \pi(6n)^{-1/2}(1 + o(1))$$

for all $k$.

Before we begin the proof we introduce some notation.

**Notation 1** Given a set $A$ of positive integers, we define $\pi_A(x)$ to be the number of elements of $A$ that are less than or equal to $x$.

## 2 Acknowledgments

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3 Proofs

We require the following straightforward lemma.

**Lemma 1** Suppose we have three power series

\[ A(x) = \sum_i \alpha_i x^i, \quad B(x) = \sum_i \beta_i x^i, \quad \text{and} \quad C(x) = \sum_i \gamma_i x^i, \]

where \( B(x) = (1 - x^d)^{-1} A(x) \) and \( C(x) = (1 - x)^{-1} B(x) \). Suppose also \( \alpha_n, \beta_n, \gamma_n \) are eventually positive and that \( \alpha_n = O(\gamma_n/n) \). Then \( \beta_n = O(\gamma_n n^{-1/2}) \).

**Proof.** Take \( N_0 \) such that \( \alpha_n, \beta_n, \gamma_n > 0 \) for all \( n \geq N_0 \). Notice that

\[ \gamma_n = \sum_{i \leq n} \beta_i \]

and hence

\[ \gamma_j \geq \gamma_k \geq 0 \text{ whenever } j \geq k \geq N_0 \] (3.1)

Choose a \( C \) such that \( \alpha_n < C\gamma_n/n \) for all \( n \) greater than \( N_0 \). Suppose \( \beta_n \neq O(\gamma_n n^{-1/2}) \). Then there exists some \( m \) such that \( \beta_m > (C+2)\gamma_m m^{-1/2} \), \( m - d\lfloor m^{1/2} \rfloor - 1 > N_0 \), and

\[ |m^{1/2}|(\lfloor m^{1/2} \rfloor + 1)/2(m - d\lfloor m^{1/2} \rfloor) < 1. \] (3.2)

Notice

\[ \alpha_m = \beta_m - \beta_{m-d} < C\gamma_m/m. \]

Hence \( \beta_{m-d} > (C+2)\gamma_m m^{-1/2} - C\gamma_m/m \). Similarly,

\[ \alpha_{m-d} = \beta_{m-d} - \beta_{m-2d} < C\gamma_{m-d}/(m - d). \]
Thus \( \beta_{m-2d} > (C+2)\gamma_m m^{-1/2} - C\gamma_m/m - C\gamma_{m-d}/(m-d) \). And by induction we have that if \( rd < m - N_0 \),

\[
\beta_{m-rd} \geq (C+2)\gamma_mm^{-1/2} - \sum_{j=0}^{r-1} C\gamma_{m-jd}/(m-jd)
\]

\[
\geq (C+2)\gamma_mm^{-1/2} - \sum_{j=0}^{r-1} C\gamma_m/(m-rd) \quad \text{(by (3.1))}
\]

\[
\geq (C+2)\gamma_mm^{-1/2} - Cr\gamma_m/(m-rd). \quad \text{(3.3)}
\]

Notice if we take \( q \) to be the greatest integer less than or equal to \( m^{1/2} \), then we have

\[
\gamma_m = \beta_m + \beta_{m-1} + \cdots + \beta_{mqid} + \gamma_{m-1}
\]

\[
\geq \sum_{i=0}^{q} \beta_{m-id}
\]

\[
\geq \sum_{i=0}^{q} \left( (C+2)\gamma_mm^{-1/2} - Ci\gamma_m/(m-id) \right) \quad \text{(by (3.3))}
\]

\[
\geq \sum_{i=0}^{q} \left( (C+2)\gamma_mm^{-1/2} - Ci\gamma_m/(m-qi) \right)
\]

\[
= (C+2)(q+1)\gamma_mm^{-1/2} - q(q+1)C\gamma_m/2(m-qi)
\]

\[
\geq (C+2)\gamma_m - C\gamma_m \quad \text{(by (3.2))}
\]

\[
= 2\gamma_m.
\]

This is a contradiction. Hence the lemma is true. \( \blacksquare \)

We now prove a proposition that will allow us to prove the Bateman-Erdős conjecture.
Proposition 1 Suppose that $A$ is a set of positive integers having property $P_k$. Then $p_A^{(k)}(n) = O(p_A^{(k-1)}(n)n^{-1/2})$.

Proof. When $A$ is finite this is proven in Theorem 3 of [1]. Hence it suffices to prove the proposition when $A$ is infinite. Bateman and Erdős showed (see Lemma 2 of [1]) that $A$ has property $P_k$ if and only if some finite subset of $A$ has property $P_k$. Hence we may choose $d, e \in A$ such that $d, e > 1$ and

$$A_1 := A - \{d\} \quad (3.4)$$

and

$$A_2 := A - \{d, e\} \quad (3.5)$$

have property $P_k$. Notice

$$(1 - x^e) (\sum_{j \geq 0} p_A^{(k)}(j)x^j) = (1 - x^e)(1 - x)^k \prod_{a \in A_1} (1 - x^a)^{-1}$$

$$= (1 - x)^k \prod_{a \in A_2} (1 - x^a)^{-1}$$

$$= \sum_{j \geq 0} p_A^{(k)}(j)x^j,$$

and hence

$$p_{A_2}^{(k)}(n) = p_{A_1}^{(k)}(n) - p_{A_1}^{(k)}(n - e), \quad (3.6)$$

where $p_{A_1}^{(k)}(j)$ is understood to be zero when $j < 0$. Let

$$H(x) = \sum_{i \geq 0} h_i x^i = \log \left( (1 - x)^k (1 - x^d) \prod_{a \in A} (1 - x^a)^{-1} \right). \quad (3.7)$$
and let

\[ S(x) = \sum_{j \geq 0} s_j x^j := x (1 - x^e)^{-1} H'(x). \quad (3.8) \]

Notice

\[ x H'(x) = -kx/(1 - x) - dx^d/(1 - x^d) + \sum_{a \in A} ax^a/(1 - x^a). \]

If we consider just the positive contribution to the coefficients of \( xH'(x) \), we obtain the following inequality.

\[
\begin{align*}
    s_n &= \sum_{j \leq n} j h_j \\
    &\leq \sum_{j \leq n} \sum_{\{a \in A : a | j\}} a \\
    &= \sum_{\{a \in A : a \leq n\}} \sum_{j \leq n/a} a \\
    &= \sum_{\{a \in A : a \leq n\}} \lfloor n/a \rfloor a \\
    &\leq \sum_{\{a \in A : a \leq n\}} n \\
    &= n \pi_A(n) \\
    &\leq n^2. \quad (3.9)
\end{align*}
\]

Similarly, if we consider only the negative contribution we find

\[
\begin{align*}
    s_n &= \sum_{j \leq n} j h_j \\
       &\leq \sum_{j \equiv n \mod b} j h_j
\end{align*}
\]
\[
\begin{align*}
\sum_{j \leq n} k - \sum_{m \leq n/d} d & \geq -kn - dn/d \\
& = -(k + 1)n. \quad (3.10)
\end{align*}
\]

Combining (3.9) and (3.10), we see

\[|s_n| \leq (k + 1)n^2. \quad (3.11)\]

From Eq. (3.7) we have

\[
\sum_{n \geq 0} p_{A_1}^{(k)}(n)x^n = \exp(H(x)). \quad (3.12)
\]

Differentiating both sides of this equation we see

\[
\begin{align*}
\sum_{i \geq 1} ip_{A_1}^{(k)}(i)x^{i-1} & = H'(x) \exp(H(x)) \\
& = H'(x) \sum_{i \geq 0} p_{A_1}^{(k)}(i)x^i \quad (\text{using (3.12)}) \\
& = S(x) \sum_{i \geq 0} (p_{A_1}^{(k)}(i) - p_{A_1}^{(k)}(i - b))x^{i-1} \quad (\text{using (3.8)}) \\
& = S(x) \sum_{i \geq 0} p_{A_2}^{(k)}(i)x^{i-1} \quad (\text{by (3.6)}). \quad (3.13)
\end{align*}
\]

Comparing the coefficients of \(x^{n-1}\) of the first and last line of (3.13) we find

\[
np_{A_1}^{(k)}(n) = \sum_{i=0}^{n} s_{n-i}p_{A_2}^{(k)}(i). \quad (3.14)
\]
Recall that $A_2$ has property $P_k$, and hence there exists some $N > 0$ such that $p_{A_2}^{(k)}(j) > 0$ for all $j \geq N$. Using this fact along with (3.9) and (3.11), we rewrite the right hand side of (3.14) as follows.

$$\sum_{i=0}^{n} s_{n-i} p_{A_2}^{(k)}(i) + \sum_{i<N} s_{n-i} p_{A_2}^{(k)}(i)$$

$$\leq \sum_{i=0}^{n} (n-i)^2 p_{A_2}^{(k)}(i) + N(k+1)n^2 \max_{i<N} |p_{A_2}^{(k)}(i)|$$

$$= \sum_{i=0}^{n} (n-i)^2 p_{A_2}^{(k)}(i) + O(n^2). \quad (3.15)$$

Thus there exists some $C > 0$ such that

$$np_{A_1}^{(k)}(n) \leq \sum_{i=0}^{n} (n-i)^2 p_{A_2}^{(k)}(i) + Cn^2. \quad (3.16)$$

From Lemma 1 of [1], we know that there exist $C_1, C_2 > 0$ such that

$$C_1n^2 < [x^n](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1} < C_2n^2. \quad (3.17)$$

for all $n$. Hence we can say that

$$C_1 np_{A_1}^{(k)}(n)$$

$$\leq \sum_{i=0}^{n} C_1(n-i)^2 p_{A_2}^{(k)}(i) + C_1 Cn^2$$

$$\leq \sum_{i=0}^{n} \left( [x^{n-i}](1-x)^{-1}(1-x^d)^{-1}(1-x^e)^{-1} \right) p_{A_2}^{(k)}(i) + C_1 Cn^2$$

10
\[
= \sum_{i=0}^{n} \left( [x^{n-i}] (1 - x)^{-1} (1 - x^{d})^{-1} (1 - x^{e})^{-1} \right) p_{A_2}^{(k)}(i) \\
- \sum_{i<N} \left( [x^{n-i}] (1 - x)^{-1} (1 - x^{d})^{-1} (1 - x^{e})^{-1} \right) p_{A_2}^{(k)}(i) + C_1 C n^2 \\
\leq \sum_{i=0}^{n} \left( [x^{n-i}] (1 - x)^{-1} (1 - x^{d})^{-1} (1 - x^{e})^{-1} \right) p_{A_2}^{(k)}(i) \\
+ NC_2 n^2 (\max_{i<N} |p_{A_2}^{(k)}(i)|) + C_1 C n^2 \\
= \sum_{i=0}^{n} \left( [x^{n-i}] (1 - x)^{-1} (1 - x^{d})^{-1} (1 - x^{e})^{-1} \right) p_{A_2}^{(k)}(i) + O(n^2). \quad (3.18)
\]

Recall that
\[
p_{A_2}^{(k)}(i) = [x^i] (1 - x)^k (1 - x^{d}) (1 - x^{e}) \prod_{a \in A} (1 - x^{a})^{-1}.
\]

Hence the last line of (3.18) can be replaced by
\[
[x^n](1 - x)^{k-1} \prod_{a \in A} (1 - x^{a})^{-1} + O(n^2) = p_{A}^{(k-1)}(n) + O(n^2).
\]

Hence there exists a \( D > 0 \) such that
\[
C_1 n p_{A_1}^{(k)}(n) \leq p_{A_1}^{(k-1)}(n) + D n^2.
\]

By Theorem 5.i. of [1] we have that \( n^2 = o(p_{A}^{(k-1)}(n)) \), hence
\[
p_{A_1}^{(k)}(n) = O(p_{A_1}^{(k-1)}(n)/n).
\]

Notice
\[
\sum_{i \geq 0} p_{A}^{(k)}(i) x^i = (1 - x^{d})^{-1} \sum_{i \geq 0} p_{A_1}^{(k)}(i) x^i.
\]
and
\[ \sum_{i \geq 0} p_A^{(k-1)}(i)x^i = (1 - x)^{-1} \sum_{i \geq 0} p_A^{(k)}(i)x^i. \]

Applying Lemma 1, taking \( \alpha_n, \beta_n \) and \( \gamma_n \) to be \( p_A^{(k)}(n), p_A^{(k)}(n) \) and \( p_A^{(k-1)}(n) \) respectively, we see that
\[ p_A^{(k)}(n) = O(p_A^{(k-1)}(n)n^{-1/2}). \]

This proves the proposition. ■

We now complete the proof of the Bateman-Erdős conjecture. To do this, we will need the following simple lemma.

Lemma 2 Suppose \( \{f_n\} \) is a sequence of integers that is eventually positive and that there exists a positive integer \( c \) such that \( f_n - f_m > 0 \) whenever \( n - m > c \). Moreover, suppose that \( f_{n-1}/f_n \to 1 \). If
\[ \sum_{i \geq 0} h_i x^i := (1 - x)(1 - x^d)^{-1}(\sum_{k \geq 0} f_k x^k), \]

then \( h_n = O(f_n) \).

Proof. Notice
\[ h_n = \sum_{0 \leq j \leq n/d} f_{n-jd} - \sum_{0 \leq j \leq (n-1)/d} f_{n-1-jd}. \]

Choose an integer \( r \) such that \( rd - 1 > c \). we will say that \( f_k = 0 \) for all \( k < 0 \). Then we have
\[ h_n = f_n + f_{n-d} + \cdots + f_{n-(r-1)d} - \sum_{0 \leq j \leq -r+n/d} (f_{n-1-jd} - f_{n-jd-rd}) \]
\[ - \sum_{-r+n/d < j \leq (n-1)/d} f_{n-1-jd} \leq f_n + f_{n-d} + \cdots + f_{n-(r-1)d} + r \max_{j \leq rd-1} |f_j| \]
\[ = f_n + f_{n-d} + \cdots + f_{n-(r-1)d} + O(1). \quad (3.19) \]

Similarly,
\[ h_n \geq -f_{n-1} - f_{n-1-d} - \cdots - f_{n-1-(r-1)d} + O(1). \quad (3.20) \]

The fact that \( \{f_n\} \) is a sequence of integers with the property that \( f_n > f_m \) whenever \( n - m > 0 \) shows that \( f_n \to \infty \). Combining this fact with (3.19) and (3.20) and the fact that \( f_{n-1}/f_n \to 1 \), we see that
\[ -r \leq \liminf_{n} h_n/f_n \leq \limsup_{n} h_n/f_n \leq r. \]

Hence \( h_n = O(f_n) \) as required.

We are finally ready to prove the conjecture of Bateman and Erdős.

**Theorem 1** Conjecture 1 is correct.

**Proof.** Suppose \( A \) has property \( P_k \). As stated in the introduction, we only need to prove the conjecture when \( A \) is infinite. Thus we assume that \( A \) is infinite. Choose \( d \in A \) such that
\[ A_1 := A - \{d\} \]
has property \( P_k \). Since \( A_1 \) has property \( P_k \), we have that \( p^{(k)}_{A_1}(n) \) is eventually positive. Moreover, by Proposition 1 we have that
\[ p^{(k)}_{A_1}(n) = O(p^{(k-1)}_{A_1} n^{-1/2}). \quad (3.21) \]
By Theorem 6 of [1], we have that there exists a $c$ such that $p_{A_1}^{(k)}(n) - p_{A_1}^{(k)}(m) > 0$ whenever $n - m > c$. Also, $p_{A_1}^{(k)}(n - 1)/p_{A_1}^{(k)}(n) \to 1$ (by Theorem 5.ii. of [1]). Furthermore, we have that

$$
\sum_{i \geq 0} p_{A}^{(k+1)}(i)x^i = (1 - x)(1 - x^d)^{-1}(\sum_{j \geq 0} p_{A_1}^{(k)}(j)x^j).
$$

Hence by an application of the preceding lemma, taking $f_n$ and $h_n$ to be $p_{A_1}^{(k)}(n)$ and $p_{A}^{(k+1)}(n)$ respectively, we see that

$$
p_{A}^{(k+1)}(n) = O(p_{A_1}^{(k)}(n)). \tag{3.22}
$$

Next notice that

$$(1 - x^d)(1 - x)^{-1}\sum_{i \geq 0} p_{A}^{(k)}(i)x^i = \sum_{j \geq 0} p_{A_1}^{(k-1)}(j)x^j.$$

Therefore

$$p_{A_1}^{(k-1)}(n) = \sum_{j=0}^{d-1} p_{A_1}^{(k)}(n - j).$$

By Theorem 5.ii. of [1], we know that for a fixed integer $j$,

$$p_{A}^{(k)}(n - j)/p_{A}^{(k)}(n) \to 1$$

as $n$ tends to infinity. Hence

$$p_{A_1}^{(k-1)}(n)/p_{A}^{(k)}(n) \to \sum_{j=0}^{d-1} 1 = d.$$
Combining this fact with (3.21) and (3.22), we see that

\[ p_A^{(k+1)}(n) = O(p_A^{(k)}(n)) = O(p_A^{(k-1)}(n)n^{-1/2}) = O(p_A^{(k)}(n)n^{-1/2}). \]

This completes the proof of the Bateman-Erdős conjecture. \( \text{\textbackslash qed} \)

References

