On the transcendence degree of subfields of division algebras

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Abstract
We study subfields of quotient division algebras of domains of finite GK dimension and introduce a combinatorial property we call the straightening property. We show that many classes of algebras have this straightening property and show that if \( A \) is a domain of GK dimension \( d \) with this property that is not PI, then the maximal subfields of the quotient division algebra of \( A \) have transcendence degree at most \( d - 1 \), proving a special case of a conjecture of Small.

1 Introduction

Given a finitely generated algebra \( A \) over a field \( F \), the Gelfand-Kirillov dimension (GK dimension for short) of \( A \) is defined to be:

\[
\text{GKdim}(A) = \limsup_{n \to \infty} \frac{\log \dim(V^n)}{\log n},
\]

where \( V \) is a finite dimensional \( F \)-vector subspace of \( A \) that contains 1 and generates \( A \) as a \( K \)-algebra. We note that this definition is independent of the choice of vector space \( V \) with the above properties. GK dimension has seen great application over the recent years due to the fact that it is a noncommutative analogue of Krull dimension; that is, if \( A \) is a finitely generated commutative algebra then the GK dimension and Krull dimension coincide. For this very reason GK dimension has been used to give noncommutative generalizations of theorems from classical algebraic geometry. We refer the reader to Krause and Lenagan [7] for the basic facts about GK dimension.

One important consequence of having finite GK dimension is that a finitely generated algebra \( A \) that is a domain of finite GK dimension has a quotient division algebra [7, Proposition 4.13]. Just as a commutative domain has a field of fractions, noncommutative domains sometimes have a quotient division algebra formed by inverting the nonzero elements. The main problem is that localization is not well-behaved in general for noncommutative rings and so one cannot in general guarantee that a noncommutative domain will

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have a quotient division algebra. Thus finite GK dimension provides a simple criterion, which when met guarantees the existence of such an object.

We study subfields of quotient division algebras of domains of finite GK dimension. There is a natural dichotomy that arises: algebras that satisfy a polynomial identity (PI algebras, for short), behave similarly to commutative algebras and can have large subfields; and algebras that do not satisfy a PI tend to have “smaller” subfields. We consider the following conjecture of Small [13].

**Conjecture 1.1.** (Small) Let $A$ be a finitely generated non-PI domain of GK dimension $d$ over a field $F$. Let $K$ be a subfield of the quotient division algebra of $A$ that contains $F$. Then $K$ has transcendence degree at most $d - 1$ over $F$.

Zhang [13, Conjecture 8.1] showed that the transcendence degree is at most $d$, and, moreover, the hypothesis that $A$ be non-PI is unnecessary with this bound. If $A$ is PI, then the quotient division algebra of $A$ is finite dimensional over its centre; if, on the other hand, $A$ is not PI then the quotient division algebra of $A$ is infinite dimensional over its centre and over any subfield. This conjecture is known for algebras of GK dimension strictly less than three; finitely generated domains of GK dimension at most one are PI [9] and behaviour of commutative subrings and subfields of domains of GK dimension less than three is relatively well-understood [1, 2, 10, 11].

We approach Conjecture 1.1 from this angle. The main difficulty lies in distinguishing between being finite dimensional over a subfield and being infinite dimensional over a subfield, but being “algebraic” over a subfield in a certain sense, which we now make precise. The main obstacle we encounter in investigating Conjecture 1.1 occurs when $D$ is left algebraic over $K$; that is, each element $x \in D$ satisfies a relation of the form

$$\sum_{i=0}^{n} \beta_i x^i = 0,$$

where $\beta_0, \ldots, \beta_n \in K$ are not all zero. We are thus able to prove the following theorem.

**Theorem 1.2.** Let $A$ be a finitely generated algebra over a field $F$ that is a domain of GK dimension $d$. Let $K$ be a subfield of the quotient division algebra of $A$ that contains $F$. If the quotient division algebra of $A$ is not left-algebraic over $K$, then the transcendence degree of $K$ over $F$ is at most $d - 1$.

In fact, we conjecture that left-algebraicity over a subfield implies left-algebraicity over the centre for quotient division algebras of finite GK dimension.

**Conjecture 1.3.** Let $D$ be the quotient division algebra of a finitely generated algebra $A$ that is a domain of finite GK dimension. Let $K$ be a maximal subfield of $D$. Suppose that $D$ is left algebraic over $K$. Then $D$ is algebraic over its centre.

We next introduce a property inspired by the Poincaré-Birkhoff-Witt theorem, which we call the straightening property. We show that if $A$ is a domain with this property and $K$ is a subfield of the quotient division algebra of $A$, then either $K$ Using the straightening property, we prove Conjecture 1.1 for several classes of algebras.
Theorem 1.4. Conjecture 1.1 holds if $A$ is a domain that satisfies any of the following:

1. $A$ is a homomorphic image of the enveloping algebra of a finite dimensional Lie algebra;
2. $A$ is a ring whose associated graded ring is commutative; for example, the Weyl algebras;
3. $A$ is a homomorphic image of a ring of quantum matrices and other quantized coordinate rings [3];
4. $A$ is a homomorphic image of a generalized down-up algebra [4];
5. $A$ is a homomorphic image of the algebras described by Witten [12].

We note that this property has been looked at in other contexts (see, for example, Jespers and Okninski [5, 6]). In Section 2, we look at Zhang’s lower transcendence degree and prove Theorem 1.2. In Section 3, we introduce the straightening property and prove Theorem 1.4.

2 Subfields of division algebras

We use Zhang’s lower transcendence degree to investigate subfields of division algebras. The following is related to Section 5 of Zhang [13]. Following Zhang, we introduce a two-sided version of his Lower transcendence degree. Given an $F$-algebra $A$, the lower transcendence degree of $A$, $\text{Ld}(A)$, is defined to be the supremum over all nonnegative real numbers $\alpha$ such that there exist a positive constant $C$ and a finite dimensional $F$-vector subspace $V$ of $A$ such that

$$\dim_F(VW) \geq \dim(W) + C (\dim(W))^\frac{\alpha-1}{\alpha}$$

for every finite dimensional $F$-vector subspace $W$ of $A$.

Given an algebra $A$, we instead consider the supremum over all nonnegative real numbers $\alpha$ such that there exists a finite dimensional subspaces $F$-vector subspaces $V$ of $A$ and a positive constant $C$ such that for every finite dimensional $F$-vector subspace $W$ of $A$ that contains 1 we have

$$\dim(VWV) \geq \dim(W) + C (\dim(W))^\frac{\alpha-1}{\alpha}.$$  

We call this quantity the two-sided lower transcendence degree of $A$ and denote it by $T\text{Ld}(A)$. We note that the condition that $1 \in W$ is not really important, but we use it to simplify some of the proofs that appear later.

We note that there are some disadvantages to working with this quantity instead of with lower transcendence degree, since it prevents one from doing any sort of inductive arguments when studying chains of division subalgebras.

We begin with an estimate that is essentially due to Zhang [13].
Proposition 2.1. Let $A$ be a finitely generated domain over a field $F$. Then

$$\text{TLd}(Q(A)) \leq \text{TLd}(A) \leq \text{GKdim}(A).$$

Proof. We first show that $\text{TLd}(A) \leq \text{GKdim}(A)$. Let $\alpha = \text{TLd}(A)$ be the two-sided lower transcendence degree of $A$ and let $\beta < \alpha$. Then there exists a finite dimensional $F$-vector spaces $V$ and a positive constant $C$ such that

$$\dim(VWV) \geq \dim(W) + C \left( \dim(W) \right)^{\frac{\beta - 1}{\beta}}$$

for every finite dimensional $F$-vector space $W$ that contains 1. Let $U$ be a finite dimensional generating subspace for $A$ containing 1, $V$. Then

$$\dim(U^{n+2}) \geq \dim(VU^nV) \geq \dim(U^n) + C \left( \dim(U^n) \right)^{\frac{\beta - 1}{\beta}}.$$ 

It follows that the GK dimension of $A$ is at least $\beta$ (cf. Zhang [13, Lemma 1.3]). Since $\beta < \alpha$ is arbitrary, we see that $A$ has GK dimension at least $\alpha$. Next suppose that $\text{TLd}(Q(A)) > \gamma$ and that $V$ is a finite dimensional $F$-vector subspace of the quotient division algebra of $A$ such that

$$\dim(VWV) \geq \dim(W) + C \left( \dim(W) \right)^{\frac{\gamma - 1}{\gamma}}$$

for every finite dimensional subspace $W$ of the quotient division algebra of $A$ that contains 1. Then we can pick $a$ and $b$ in $A$ such that $V_1' = aV$ and $V_2' = Vb$ are subspaces of $A$ and then select $V'$ containing $V_1'$ and $V_2'$. Then since $A$ lies in the quotient division algebra of $A$, we have

$$\dim(V'WV') \geq \dim(W) + C \left( \dim(W) \right)^{\frac{\gamma - 1}{\gamma}}$$

for every finite dimensional $F$-vector subspace $W$ of $A$ that contains 1. Hence $\text{TLd}(A) > \gamma$. Thus

$$\text{TLd}(A) \geq \text{TLd}(Q(A)).$$

Definition. Let $A$ be an algebra containing a divisional algebra $D$. We say that $A$ is left algebraic over $D$ if for each $a \in A$ there exist a natural number $n$ and $\beta_0, \ldots, \beta_n$ not all zero such that

$$\beta_n a^n + \beta_{n-1} a^{n-1} + \cdots + \beta_0 = 0.$$ 

Remark 1. Let $A$ be an algebra containing a division algebra $D$. If $A$ is not left algebraic over $D$ then there exists some $x \in A$ such that

$$\sum_{n=0}^{\infty} D x^n$$

is direct.
We now give two estimates that we will use to estimate the transcendence degree of subfields.

**Lemma 2.2.** Let $D$ be a division algebra with a (not necessarily central) subfield $K$. Suppose $W$ is a finite dimensional $K$-vector subspace of $D$ and $x \in D$ are such that:

1. $W$ has a decomposition 
   $$W = U_0 \oplus \cdots \oplus U_m,$$
   where $U_0, \ldots, U_m$ are finite dimensional $K$-vector spaces;
2. the sum $U_0 + \cdots + U_m$ is direct;
3. $U_i x \cap (U_0 + \cdots + U_i) = (0)$ for $i \leq m$;
4. $U_i x \subseteq U_1 + \cdots + U_{i+1}$ for $i < m$.

Then 
$$\dim(W + Wx) \geq \dim(W) + \max_{0 \leq i \leq m} \dim(U_i).$$

**Proof.** Since we prove this by induction on $m$. If $m = 0$, $Wx + W$ is direct and hence
$$\dim(Wx + W) = \dim(W) = \dim(U_0),$$
and so the claim is true when $m = 0$. Now assume that the claim is true for all integers $< m$. Let $W_1 = U_0 + \cdots + U_{m-1}$. Then
$$\dim(W_1 x + W_1) - \dim(W_1) \geq \max_{i < m} \dim(U_i).$$

Finally note that
$$W + Wx \supseteq (W_1 + W_1 x) \oplus U_m x$$
and hence
$$\dim(W + Wx) \geq \max_{i < m} \dim(U_i) + \dim(W_1) + \dim(U_m).$$

Since
$$W_1 \oplus U_m = W,$$
we see that
$$\dim(W + Wx) - \dim(W) \geq \max_{i < m} \dim(U_i).$$

On the other hand,
$$W + Wx \supseteq W \oplus U_m x,$$
and so
$$\dim(W + Wx) - \dim(W) \geq \dim(U_m).$$

The result now follows.

**Lemma 2.3.** Let $D$ be a division algebra with central subfield $F$ and let $K$ be a subfield of $D$ that is an extension of $F$ of transcendence degree $d$. Suppose $W$ is a finite dimensional $F$ vector subspaces of $D$ and $x \in A$ are such that:
1. there exist subspaces $U_0, \ldots, U_m$ such that $U_0 \oplus \cdots \oplus U_m = W$ for $0 \leq i \leq d$ and the sum

$KU_0 + \cdots + KU_m$

is direct;

2. $V$ is a finite dimensional $F$-vector subspace of $K$ that contains 1 and contains $d$ algebraically independent elements (over $F$) of $K$.

Then there is some positive constant $C > 0$ such that

$$\dim_F(VW) \geq \dim_F(W) + \sum_{i=0}^{m} C (\dim_F(W_{i+1}/W_i))^{\frac{d-1}{d}}$$

for every finite dimensional $F$-vector subspace $W$ of $D$.

**Proof.** We let

$$a_i := \dim_F(W_i/W_{i-1}) = \dim_F(U_i).$$

a result of Zhang [13, Cf. Theorem 2.7], there is some positive constant $C$ such that

$$\dim_F(VU) \geq \dim_F(U) + C (\dim_F(U))^{(d-1)/d}$$

for every finite dimensional $F$-vector subspace $U$ of $K$. Thus

$$\dim_F(VU_i) \geq \dim_F(U_i) + Ca_i^{(d-1)/d}$$

for all $i$, and so

$$\dim_F(VW) - \dim_F(W) \geq \sum_{i=0}^{m} \dim_F(VU_i/U_i)$$

Therefore:

$$\geq \sum_{i=0}^{m} Ca_i^{(d-1)/d}.$$
Proof. Without loss of generality, we may assume that
\[ a_0 \geq a_1 \geq \cdots \geq a_m. \]
Suppose that
\[ a_0 \leq \frac{(d-1)^d}{d^d} N^d/(d+1). \]
By the mean value theorem
\[ a_i^{(d-1)/d} - a_{i+1}^{(d-1)/d} \geq (a_i - a_{i+1}) \frac{d-1}{d} a_i^{-1/d} \geq (a_i - a_{i+1}) \frac{d-1}{d} \frac{d}{d-1} N^{-1/(d+1)}. \]
Thus
\[
\sum_{i=0}^{m} a_i^{(d-1)/d} = (m+1)a_m^{(d-1)/d} + \sum_{i=0}^{m-1} (i+1)(a_i^{(d-1)/d} - a_{i+1}^{(d-1)/d}) \\
\geq (m+1)a_m^{(d-1)/d} + \sum_{i=0}^{m-1} (i+1)(a_i - a_{i+1}) \frac{d-1}{d} \frac{d}{d-1} N^{-1/(d+1)} \\
= (m+1)a_m^{(d-1)/d} + N^{-1/(d+1)}(a_0 + \cdots + a_m - (m+1)a_m) \\
= (m+1)a_m^{(d-1)/d} + N^{d/(d+1)} - (m+1)a_m N^{-1/(d+1)} \\
= N^{d/(d+1)} + (m+1)a_m^{(d-1)/d} \left(1 - (a_m N^{-d/(d+1)})^{1/d}\right) \\
\geq N^{d/(d+1)}. \]
The result follows. \(\square\)

We are almost ready to prove Theorem 1.2, we need a lemma about selecting bases for vector spaces.

Lemma 2.5. Let \( D \) be a division algebra with central subfield \( F \) and let \( K \) be a subfield of \( D \) containing \( F \). Suppose that \( x \in D \) is such that
\[
\sum Kx^i
\]
is direct and suppose that \( W \) is a finite dimensional \( F \)-vector subspace of \( D \) that contains 1. Then there exists a basis \( \{x_1, \ldots, x_p\} \) for \( KW \) with the properties:

1. there exist \( 0 = j_0 < j_1 < \cdots < j_m < j_{m+1} = p \) such that for \( i < m \) and \( j_{i-1} < \ell \leq j_i \),
   \( x_\ell x \in Kx_{j_1} + Kx_{j_2} + \cdots + Kx_{j_{i+1}} \);
2. \( x_\ell x \not\in Kx_{j_1} + \cdots + Kx_{j_i} \) for \( j_{i-1} \leq \ell \leq j_i \);
3. \( x_\ell x \not\in W \) for \( \ell > j_m \).
**Proof.** Suppose this result is not true. Then let $W$ be an $F$-vector space of minimal dimension for which the conclusion of the statement of the lemma fails. Let

$$W_1 = \{ w \in W \mid wx \in W \}.$$  

Notice that if the dimension of $W_1$ is strictly smaller than the dimension of $W$, then we are done, since by minimality of the dimension of $W$, $W_1$ must have a basis $\{x_1, \ldots, x_m\}$ for $W$ with the properties:

1. there exist $1 = j_0 < j_1 < \cdots < j_{m-1} < j_m = p$ such that for $i < m-1$ and $j_{i-1} < \ell \leq j_i$, $x_\ell x \in Kx_1 + Kx_2 + \cdots + Kx_{j_{i+1}}$;
2. $x_\ell x \not\in Kx_1 + \cdots + Kx_{j_i}$ for $j_{i-1} \leq \ell < j_i$;
3. $x_\ell x \not\in KW$ for $\ell > j_{m-1}$.

Now, we let $x_{p+1}, \ldots, x_{p+d}$ be elements of $W$ such that their images in $W/W_1$ form a basis for the quotient space. Then taking $j_{m+1} = p + d$, we see that $\{x_1, \ldots, x_{p+d}\}$ is a basis for $W$ satisfying the conditions. Hence we may assume that $Wx = W$. But this gives

$$\sum_{i \geq 0} Wx^i = W.$$  

But $W$ contains $K$ since $1 \in W$ and hence

$$\sum_{i \geq 0} Kx^i \subseteq W,$$

contradicting the fact that the sum is direct. \qed

Finally, need is a remark due to Zhang, which essentially allows us to filter a vector space in a useful way when estimating dimensions.

**Remark 2.** Let $B$ be a subalgebra of an algebra $A$, and suppose that $A$ is a free left $B$-module. Suppose that $W \subseteq \bigoplus_{i=1}^m Bx_i \subseteq A$ is a finite dimensional $F$-vector space. Then there exist finite $F$-linearly independent sets $S_1, \ldots, S_m$ of $B$ such that for every $s \in S_i$ there exists some element $\alpha_s \in \bigoplus_{j=i+1}^m Bx_j$ such that

$$\bigcup_{i=1}^m \bigcup_{s \in S_i} \{ sx_i + \alpha_s \}.$$  

**Proof.** Cf. Zhang [13, Proof of Theorem 2.4 (1)]. \qed

**Theorem 2.6.** Let $D$ be a division algebra with central subfield $F$ and let $K$ be a subfield of $D$ that contains $F$. Then

$$\text{TLd}(D) \geq \text{trdeg}_F K + 1.$$
**Proof.** Suppose \( D \) is not left algebraic over \( K \). Then there exists some \( x \in D \) such that \( \sum Kx^i \) is direct. We take
\[
V_2 = F + Fx.
\]
(3)

Let \( t_1, \ldots, t_d \) be algebraically independent elements of \( K \) over \( F \). We take
\[
V_1 = Fy_1 + \cdots + Fy_d + F.
\]
(4)

Let \( W \) be a finite dimensional \( F \)-vector subspace of \( D \). Consider the left \( K \)-vector space \( KW \). By Lemma 2.5, we can pick a \( K \)-basis \( \{x_1, \ldots, x_p\} \) for \( KW \) with the properties:

1. there exist \( 0 = j_0 < j_1 < \cdots < j_m < j_{m+1} = p \) such that for \( i < m \) and \( j_{i-1} < \ell \leq j_i \),
   \[ x_\ell x \in Kx_1 + Kx_2 + \cdots + Kx_{j_{i+1}}; \]
2. \( x_\ell x \not\in Kx_1 + \cdots + Kx_{j_i} \) for \( j_{i-1} \leq \ell \leq j_i \);
3. \( x_\ell x \not\in KW \) for \( \ell > j_m \).

By Remark 2 there exist finite \( F \)-linearly independent sets \( S_1, \ldots, S_p \) of \( K \) such that for every \( s \in S_i \) there exists some element \( \alpha_s \in \bigoplus_{j=0}^{j_i-1} Kx_j \) such that
\[
\bigcup_{i=1}^{m} \bigcup_{s \in S_i} \{sx_i + \alpha_s\}
\]
is a basis for \( W \). We let \( U_i \) be the \( F \)-vector space spanned by
\[
\bigcup_{i=j_{i+1}}^{j_i+1} \bigcup_{s \in S_i} \{sx_i + \alpha_s\}
\]
Then we note that
\[
KU_0 + \cdots + KU_m
\]
is direct. By Lemma 2.3 we have
\[
\dim_F(V_1W) - \dim_F(W) \geq \sum_{i=1}^{m} (\dim_F(U_i))^{(d-1)/d}. \tag{5}
\]

Similarly,
\[
\dim_F(WV_2) - \dim_F(W) \geq \max_i \dim_F(U_i), \tag{6}
\]
by Lemma 2.2, since \( U_i x \in KU_{i+1} + \cdots + KU_0 \) and \( U_i x \cap (KU_i + \cdots + KU_0) = (0) \). We have
\[
\dim_F(W) = \sum_i \dim_F(U_i),
\]
and hence taking \( a_i = \dim(U_i) \), we see by Lemma 2.4 that either
\[
\sum_{i=1}^{m} a_i^{(d-1)/d} \geq \left(\dim_F(W)\right)^{d/(d+1)}
\]
or
\[ \max_i a_i \geq \frac{(d - 1)^d}{d^d} (\dim_F(W))^{d/(d+1)}. \]

Either way, we see that
\[ \dim_F(V_1WV_2) - \dim_F(W) \geq \frac{(d - 1)^d}{d^d} (\dim_F(W))^{d/(d+1)} \]

and hence \( \text{TLd}(D) \geq d + 1. \)

**Proof of Theorem 1.4.** Suppose the quotient division algebra of \( A \) is not algebraic over \( K \). By Zhang’s theorem [13, Corollary 0.8], the transcendence degree of \( K \) is at most \( d \). Let \( e \) denote the transcendence degree of \( K \) over \( F \). By Theorem 2.6, \( \text{TLd}(Q(A)) \) is at least \( e + 1 \). But
\[ e + 1 \leq \text{TLd}(Q(A)) \leq \text{TLd}(A) \leq \text{GKdim}(A) = d, \]

by Proposition 2.1 and so we obtain the desired result.

\[ \square \]

## 3 Algebraicity

In this section, we investigate when a division algebra can be left algebraic over a maximal subfield. To work towards this conjecture we introduce the notion of the *straightening property*.

**Definition.** Let \( F \) be a field and let \( A \) be a finitely generated \( F \)-algebra of GK dimension \( d \). We say that \( A \) has the straightening property if there exist \( x_1, \ldots, x_m \in A \) such that the set
\[ S = \{ x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m} \mid i_1, \ldots, i_d \geq 0 \} \]

spans \( A \) as an \( F \)-vector space. We call the set \( S \) a straightening spanning set for \( A \).

This property is inspired by the Poincaré-Birkhoff-Witt theorem, which gives a basis for the enveloping algebra of a Lie algebra.

**Proposition 3.1.** The following algebras have the straightening property:

1. a homomorphic image of an algebra with the straightening property;
2. the enveloping algebra of a finite dimensional Lie algebra;
3. a domain with an associated graded ring with the straightening property;
4. the tensor product of two domains with the straightening property;
5. a finitely generated commutative algebra.
Proof. Notice if $A$ has a straightening spanning set $S$, then the image of $S$ under a surjective homomorphism is again a straightening spanning set for a homomorphic image of $A$.

The fact that an enveloping algebra of a finite dimensional Lie algebra has the straightening property is an easy consequence of the Poincaré-Birkhoff-Witt theorem [8]. Algebras that have a basis given by a straightening spanning set are often called PBW algebras.

Note that if $A$ has an associated graded ring with the straightening property, then a straightening spanning set for the graded algebra can be pulled back to give a straightening spanning set for $A$.

If $S$ and $T$ are straightening spanning sets for two algebras $A$ and $B$ then $S \otimes T$ is a straightening spanning set for $A \otimes B$.

It is straightforward to verify that a polynomial ring has the straightening property. Since every finitely generated commutative algebra is a homomorphic image of a polynomial ring, the result follows.

Lemma 3.2. Let $D$ be a division algebra that is left algebraic over a subfield $K$. If $x_1, \ldots, x_m$ are elements of $D$, then

$$
\sum_{i_1, \ldots, i_m \geq 0} K x_1^{i_1} \cdots x_m^{i_m}
$$

is a finite dimensional left $K$-vector space.

Proof. We prove this by induction on $m$. When $m = 1$, the claim just follows from the fact that $D$ is left algebraic over $K$. Now suppose the claim is true for all integers less than $m$. Then

$$
\sum_{i_1, \ldots, i_{m-1} \geq 0} K x_1^{i_1} \cdots x_{m-1}^{i_{m-1}}
$$

is a finite dimensional left $K$-vector space $W$. Write $W = Ku_1 + \cdots + Ku_d$. We note that for each $i$,

$$
\sum_j K(u_i x_m u_i^{-1})^j = \left( \sum_j Ku_i x_m^j \right) u_i^{-1}
$$

is not direct, since $D$ is left algebraic over $K$. Consequently, there is some $N$ such that for each $i \leq d$,

$$
\sum_{j \geq 0} Ku_i x_m^j = \sum_{0 \leq j \leq N} Ku_i x_m^j.
$$

Thus

$$
\sum_{i_1, \ldots, i_m \geq 0} K x_1^{i_1} \cdots x_m^{i_m} = \sum_{j=0}^N W x_m^j,
$$

is finite dimensional as a left $K$-vector space.
Proposition 3.3. Let $A$ be a finitely generated domain of integer GK dimension $d$ over a field $F$ and suppose that $A$ has the straightening property. If $K$ is a subfield of the quotient division algebra of $A$ and the quotient division algebra $D$ of $A$ is left algebraic over $K$ then $D$ is finite dimensional over $K$ as a left $K$-vector space.

Proof. By assumption there exist $x_1, \ldots, x_m \in A$ such that the set

$$S = \{x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m} \mid i_1, \ldots, i_m \geq 0\}$$

is an $F$-basis for $A$. Suppose that the quotient division algebra of $A$ is left algebraic over a subfield $K$. Consider the $K$-vector space $W = KS$. We note that $W$ is finite dimensional over $K$ by Lemma 3.2. Also, $KA = W$ and hence $KA$ is finite dimensional over $K$. Suppose that $\dim_K(KA) = e$. We claim that $D$ is $e$ dimensional as a left $K$-vector space. If not, there exist $\alpha_1, \ldots, \alpha_{e+1} \in D$ that are left-linearly independent over $K$. There exists some $b \in A$ such that $a_i := \alpha_i b \in A$ for $1 \leq i \leq m+1$. Then by construction, $Ka_1 + \cdots + Ka_{m+1}$ is direct, contradicting the fact that $\dim_K(KA)$ is finite dimensional. Hence $D$ is finite dimensional as a left $K$-vector space. □

Proof of Theorem 1.4. This follows from Theorem 1.2, Proposition 3.3, and Proposition 3.1.

4 Conclusion remark

We make the remark that the results we give can all be expressed more generally using prime Goldie rings instead of domains.

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