\[ \frac{dX}{dt} = a(X) \quad X(t_0) = x_0 \quad (1) \]

\[ X \in \mathbb{R}^n \quad a : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (2) \]

Fact: If \( a \) is locally Lipschitz, then there is a unique local solution (That is, there is some interval of time \((t_0 + \varepsilon, t_0 + \varepsilon)\) over which we can find a solution to the eqn.

1 Locally Lipschitz

On every bounded \( U \subset \mathbb{R} \), there is a const \( L_U \) such that

\[ \|a(X) - a(y)\|_2 \leq L_U \|x - y\|_2 \forall x, y \in U \quad (3) \]

Example: \( a(x) = \sqrt{x} \) is not locally Lipschitz.

2 Global/Local solution

A global solution means that we can find \( X(t) \) for all time \( t \) that solves eqn.

Weird thing: if \( a \) is locally lipschitz then for any \( t_0, x_0 \), I can find a local solution to

\[ \frac{dX}{dt} \quad (4) \]

that goes through \((t_0, x_0)\) why can’t I extend solutions indefinitely?

If \( X(t) \rightarrow \infty \) as \( t \rightarrow t_c \), then you cannot extend the solution past \( t_c \) so you don’t have a global solution.

Fact: \( a \) is locally Lipschitz then

\[ \frac{dX}{dt} = a(X) \quad X(t_0) = x_0 \quad (5) \]

either has a solution \( X(t) \) for all time \( t > t_0 \) or \( X(t) \rightarrow \pm \infty \) as \( t \rightarrow t_c > t_0 \).

3 Note

consider

\[ \frac{dX}{dt} = X \quad X(0) = 1 \quad (6) \]

\[ X(t) = e^t \quad (7) \]

\[ X(t) \rightarrow \infty \text{ only as } t \rightarrow \infty \quad (8) \]

That is not blowup.

3.1 example

\[ \frac{dX}{dt} = X^2 \quad X(0) = 1 \quad (9) \]

\[ \int x^{-2} \, dx = \int dt \quad (10) \]

\[ -x^{-1} = t + c \quad (11) \]

\[ x(0) = 1 \Rightarrow c = -1 \quad (12) \]

\[ x = \frac{1}{1 - t} \quad (13) \]
4 Can we guarantee no blowup

Easiest condition: if we have a globally Lipschitz then global solutions exist. But most interesting nonlinear choices of $a$ are not locally Lipschitz.

4.1 example

$$\frac{dX}{dt} = -V'(x) \quad V(x) = \frac{1}{4}X^4 - \frac{1}{2}X^2$$

$$a(x) = -x^3 + x$$

(14)

(15)

can check $a$ is locally Lipschitz, but is not globally Lipschitz

$$a'(x) = -3x^2 + 1$$

(16)

which can become arbitrarily large.

Eq. (17) doesn’t blow up, however $x$

$$\frac{dX}{dt} = +V'(x)$$

(17)

Will blow up

In practice: if your system is “stable” and you choose a “good” model, then your equations will have global solutions.

5 SDEs

$$dX = a(X)dt + b(X)dB$$

$$a : \mathbb{R}^n \mapsto \mathbb{R}^n$$

$$b : \mathbb{R}^n \mapsto \mathbb{R}^{n \times r}$$

(18)

(19)

(20)

Facts: If $a$ and $b$ are globally Lipschitz then there are unique solutions for all time (Global solutions). If $a$ and $b$ are locally Lipschitz, then unique local solutions and either there is blowup in finite time or there are global solutions for all trajectories.

6 Example

$$dX = -\gamma X dt + \sigma dB$$

$$a(X) = -\gamma X \quad b(X) = \sigma$$

(22)

(23)

$b$ is definitely locally Lipschitz and

$$\|a(x) - a(y)\|_2 = \| -\gamma x + \gamma y\|_2$$

$$= |\gamma| \|x - y\|_2$$

(24)

(25)

so its $a(x)$ is globally Lipschitz.

IF $a$ has a bounded derivative then it is globally Lipschitz

Proof: In 1d suppose

$$a' \leq \forall x$$

(26)

Then

$$a(x) - a(y) = (x - y)a'(x)$$

$$\|a(x) - a(y)\|_2 = \|x - y\|_2 |a'(z)|$$

(27)

(28)

If the coefficients are linear or linear+const.