

1 Neville's Method

Neville's method can be applied in the situation that we want to interpolate $f(x)$ at a given point $x = p$ with increasingly higher order Lagrange interpolation polynomials.

For concreteness, consider three distinct points x_0, x_1 , and x_2 at which we can evaluate $f(x)$ exactly $f(x_0), f(x_1), f(x_2)$. From each of these three points we can construct an order zero (constant) "polynomial" to approximate $f(p)$

$$f(p) \approx P_0(p) = f(x_0) \tag{1}$$

$$f(p) \approx P_2(p) = f(x_1) \tag{2}$$

$$f(p) \approx P_3(p) = f(x_2) \tag{3}$$

$$\tag{4}$$

Of course this isn't a very good approximation so we turn to first order Lagrange polynomials

$$f(p) \approx P_{0,1}(p) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \tag{5}$$

$$f(p) \approx P_{1,2}(p) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2) \tag{6}$$

There is also $P_{0,2}$, but we won't concern ourselves with that one.

If we note that $f(x_i) = P_i(x)$, we find

$$P_{0,1}(p) = \frac{x - x_1}{x_0 - x_1} P_0(p) + \frac{x - x_0}{x_1 - x_0} P_1(p) \tag{7}$$

$$= \frac{(x - x_1)P_0(p) - (x - x_0)P_1(p)}{x_0 - x_1} \tag{8}$$

and similarly

$$P_{0,1}(p) = \frac{(x - x_2)P_1(p) - (x - x_1)P_2(p)}{x_1 - x_2} \tag{9}$$

In general we want to multiply $P_i(x)$ by $(x - x_j)$ where $j \neq i$ (i.e., x_j is a point that is NOT interpolated by $P_i(x)$). We take the difference of two such products and divide by the difference between the added points.

The result is a polynomial $P_{i,i-1}$ of one degree higher than either of the two used to construct it and that interpolates all the points of the two constructing polynomials combined.

This idea can be extended to construct the third order polynomial $P_{0,1,2}$

$$P_{0,1,2}(p) = \frac{(p - x_2)P_{0,1}(p) - (p - x_0)P_{1,2}(p)}{x_0 - x_2} \tag{10}$$

A little algebra will convince you that

$$P_{0,1,2}(p) = \frac{(p - x_1)(p - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(p - x_0)(p - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(p - x_0)(p - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \tag{11}$$

which is just the 3^{rd} order Lagrange polynomial interpolating the points x_0, x_1, x_2 . This shouldn't surprise you since this is the unique third order polynomial interpolating these three points.

2 Example

We are given the function

$$f(x) = \frac{1}{x} \tag{12}$$

We want to approximate the value $f(3)$.

First we evaluate the function at the three points

$$\begin{array}{ccc}
 i & x_i & f(x_i) \\
 0 & 2 & 0.5 \\
 1 & 2.5 & 0.4 \\
 2 & 4 & 0.25
 \end{array} \tag{13}$$

We can first make three separate zero-order approximations

$$f(3) \approx P_0(3) = f(x_0) = 0.5 \tag{14}$$

$$f(3) \approx P_1(3) = f(x_1) = 0.4 \tag{15}$$

$$f(3) \approx P_2(3) = f(x_2) = 0.25 \tag{16}$$

From these we proceed to construct $P_{0,1}$ and $P_{1,2}$ by using the Neville formula

$$f(3) \approx P_{0,1}(3) = \frac{(3-x_1)P_0(3) - (3-x_0)P_1(3)}{x_0 - x_1} = \frac{(3-2.5)0.5 - (3-2)0.4}{2-2.5} = 0.3 \tag{17}$$

$$f(3) \approx P_{1,2}(3) = \frac{(3-x_2)P_1(3) - (3-x_1)P_2(3)}{x_1 - x_2} = \frac{(3-4)0.4 - (3-2.5)0.25}{2.5-4} = 0.35 \tag{18}$$

So we can add these numbers to our table

$$\begin{array}{cccc}
 i & x_i & P_i & P_{i,i-1} \\
 0 & 2 & 0.5 & \\
 1 & 2.5 & 0.4 & 0.3 \\
 2 & 4 & 0.25 & 0.35
 \end{array} \tag{19}$$

Finally we can compute $P_{0,1,2}$ using $P_{0,1}$ and $P_{1,2}$

$$f(3) \approx P_{0,1,2}(3) = \frac{(3-x_2)P_{0,1}(3) - (3-x_0)P_{1,2}(3)}{x_0 - x_2} = \frac{(3-4)(0.3) - (3-2)0.35}{4-2} = 0.325 \tag{20}$$

$$\begin{array}{ccccc}
 i & x_i & P_i & P_{i,i-1} & P_{i,i-1,i-2} \\
 0 & 2 & 0.5 & & \\
 1 & 2.5 & 0.4 & 0.3 & \\
 2 & 4 & 0.25 & 0.35 & 0.325
 \end{array} \tag{21}$$

If you find yourself in the unusual situation that you know $P_{0,1,2}$, and one of $P_{0,1}$, or $P_{1,2}$, but not the other, you can always rearrange Eq. 20 to suit your purposes.