

The Riemann Hypothesis: A Resource for the Afficianado and Virtuoso Alike

by Peter Borwein, Stephen Choi, Brendan Rooney, and
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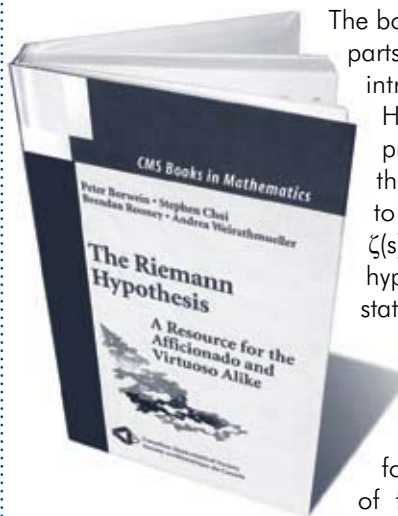
Reviewed by Jeff Hooper,
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The Riemann Hypothesis is viewed by many as the Holy Grail of mathematics. Of the seven Millennium Prize problems, it is the only one which also appeared on Hilbert's list of 23 problems a century earlier. Hilbert is in fact supposed to have said that if he were to awaken after having slept for a thousand years, his first question would be: "Has the Riemann Hypothesis been proven?"

The announcement of the Millennium Prizes a few years ago thrust the Riemann Hypothesis into the public consciousness. During the last several years a number of "popular" books have appeared on the topic of the Riemann Hypothesis: Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, du Sautoy's *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*, Rockmore's *Stalking the Riemann Hypothesis*, and Sabbagh's two books, *Dr Riemann's zeros* and *The Riemann Hypothesis: The Greatest Unsolved Problem of Mathematics*. While it's a delight to see mathematics promoted more widely, the depth of the mathematical content in these books is, as one would expect, fairly limited. For a professional mathematician or graduate student who wishes to take on a more serious study of the mathematics surrounding the Riemann Hypothesis, these popular texts can at best only scratch the surface or provide a cultural backdrop.

So where should someone turn for a more thorough mathematical treatment? There are several excellent books available on the Riemann zeta function, including Titchmarsh's classic *The Theory of the Riemann Zeta Function*, Edwards's *Riemann's Zeta Function*, Patterson's *An introduction to the Theory of the Riemann Zeta Function*, and Ivic's comprehensive *The Riemann Zeta Function*. However, in each of these the Riemann zeta function itself is the focus and the Riemann Hypothesis, though pervasive, is only brought into the foreground when appropriate.

The book under review here gives a new alternative: its focus is squarely on being an advanced introduction to the circle of ideas surrounding the Riemann Hypothesis. As such, it fills an important niche in the literature. It is mainly a source book, and the majority of its content consists of reprints of original papers and expository articles. The emphasis of the book is made clear by the authors at the outset: their choice of papers is informed by the intimate connections between the Riemann Hypothesis and the Prime Number Theorem.



The book is divided into three main parts. The first is an extensive introduction to the Riemann Hypothesis. After some analytic prerequisites are considered, there are chapters devoted to algorithms for calculating $\zeta(s)$, empirical evidence for the hypothesis, and a number of statements which are equivalent to it. Also included are discussions of various extensions of the Riemann Hypothesis, of results which follow on from its validity, and of failed attempts at a proof.

The introduction concludes with a section of formulas involving zeta values, as well as a timeline of highlights in the development of the theory through 2004. The importance of this first part should not be underestimated: there is a wealth of material here, much of which either does not appear in the texts mentioned above or can be difficult to find.

Part two consists of four 'Expert Witness' accounts of the Riemann Hypothesis: these are reprints of four recent surveys of the topic, including Bombieri's Millennium Prize Problem Description, Sarnak's 2004 elaboration, the excellent 2003 AMS Notices article by Conrey, and a 2003 article by Ivic, in which he provides and analyzes arguments which cast doubt on the validity of the Riemann Hypothesis.

The third and final part of the book is a collection of original papers on various aspects related to the Riemann Hypothesis. Each paper is preceded by a discussion to help place its results in context. The collection of papers of course includes Riemann's original 1859 paper, both in digitized form and in a translation by D.R. Wilkins, as well as Weil's 1941 paper in which he proves the Riemann Hypothesis for function fields. Each of the remaining papers either involves direct connections with the Riemann Hypothesis, or is connected to related analytic aspects, especially the Prime Number Theorem. Indeed, the papers of Hadamard and de la Vallée Poussin are both here, as is the earlier important contribution of Chebyshev. As we might expect, we also find contributions by Hardy and Littlewood: the 1914 paper in which Hardy first proves that there are infinitely many zeros of $\zeta(s)$ on the critical line, a 1915 Hardy expository article on the Prime Number Theorem, and the 1915 Hardy-Littlewood paper in which they offer new approaches to proving PNT.

Among the remaining papers are several which offer various elementary approaches to the Prime Number Theorem: the pair of independent 1949 papers by Selberg and Erdős, along with the three approaches given in the papers of Newman (1980), Daboussi (1984), and Hildebrand (1986). We also find important contributions of Turán and Skewes, as well as the disproof of Polya's Conjecture by Haselgrove. Two further

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papers of Montgomery (one co-authored with Goldston) examining differences between imaginary parts of zeros of $\zeta(s)$ provide an early connection with random matrices. The recent paper of Agrawal, Katyal, and Saxena is also included, in which the authors present their deterministic polynomial-time algorithm for determining primes.

My only real complaint with the book is the authors' focus on the purely analytic number theoretic connections with the Riemann Hypothesis, and their decision to essentially ignore other approaches. The Riemann Hypothesis remains wide open, and at present it isn't at all clear from which direction a potential proof might come. In their defence, widening the scope of the book to include other approaches and points-of-view would have demanded the inclusion of numerous papers, along with a number of their precursors, and almost certainly would have seen the book's length become unwieldy. But I would have

preferred to see, say, a compromise: a final section including major papers exploring alternative connections with the Riemann Hypothesis, such as for instance Connes' 1999 paper in which he describes connections between noncommutative geometry and the zeros of $\zeta(s)$, or the more recent paper by Conrey, Farmer, Keating, Rubinstein, and Snaith on integral moments of L -functions.

My quibbling aside, this is an excellent book, and I highly recommend it. It could be used for a graduate course or a seminar, alone or as a complement to one of the texts mentioned earlier. I hope that more books of this sort will appear in the future.

NEWS FROM DEPARTMENTS / NOUVELLES DE DÉPARTMENTS

McGill University, Montreal, QC

Promotions: D. Jakobson (Full Professor, June 1, 2008); Nilima Nigam (Associate Professor, June 1, 2008); Adrian Vetta (Associate Professor, June 1, 2008).

Appointments: Tibor Szabo (Associate Professor, August 21, 2008, Discrete Mathematics Continuous Optimization).

Retirements (rank, date): Georg Schmidt (Professor, July 31, 2008).

Resignations: Thomas Wihler (Assistant Professor, June 30, 2008).

Death: Arwel Evans (Associate Professor (post retirement), August 17, 2008).

Awards/Distinctions: Dmitry Jakobson (CMS 2008 G. de B. Robinson Award); Pengfei Guan (Royal Society of Canada Fellow, 2008); Henri Darmon (2008 John L. Synge Prize by the Royal Society of Canada); Mathew Greenberg (CMS 2008 Doctoral Prize).

University of British Columbia, Vancouver, BC

Promotions: Jimmy Feng (Professor, July 1, 2008); Ian Frigaard (Professor, July 1, 2008); Nike Vatsal (Professor, July 1, 2008).

Appointments: Omer Angel (Assistant Professor, July 1, 2008, Probability); Christoph Hauert (Assistant Professor, July 1, 2008, Mathematical Biology); Mahta Khosravi (Assistant Professor, July 1, 2008, Analysis); Young-Heon Kim (Assistant Professor, July 1, 2008, Partial Differential Equations); Akos Magyar (Assistant Professor, July 1, 2008, Analysis).

Resignations: Antoine Mellet; Ulrich Horst.

Awards/Distinctions: Ivar Ekeland (elected to the Royal Society of Canada, 2008); Martin Barlow, Michael Doebeli and Leah Keshet (have been appointed as Distinguished Scholars in Residence for 2009 at UBC's Peter Wall Institute for Advance Study); Stephanie Van Willigenburg (Humboldt Research Fellowship for 2008-2009).

University of Saskatchewan, Saskatoon, SK

Appointments: Alexei Cheviakov (Assistant Professor, July 1, 2008, Applied Mathematics); Juxin Liu, (Assistant Professor, July 1, 2008, Statistics); Ebrahim Samei (Assistant Professor, July 1, 2008, Functional Analysis, Harmonic Analysis).

Visitors: Maxime Fevrier (France, Probability-random matrix, September 20/08 to November 20/08); Fereidoun Gharamani (Canada, Mathematics-Functional Analysis, Banach Algebras, September 22/08 to October 3/08); Igor Klep (University of California - San Diego, Real Algebra, October 7-21, 2008); Michael Matusinki, France, Differential Algebra, March 17/08 to September 17/08); Katarzyna Osiak (Poland, Real Algebra, August 26/08 to October 7/08); Marina Tvalavadze (Canada, Nonassociative Algebra, September 1, 2008 to May 1, 2009); Victor Vinnikov (Israel, Analysis, Sept 23/08 to Nov 15/08); Guichang Zhang (China, Mathematics - Queue Theory, Nov 1/07 to March 31/09)

Other News: On July 1, 2008, the University of Saskatchewan officially became a full member of PIMS (Pacific Institute for the Mathematical Sciences).

Wilfrid Laurier University, Waterloo, ON

Promotions: Yongzeng (George) Lai (Associate Professor, July 1, 2008)

Appointments: Anne-Marie Allison (Assistant Professor, Applied Mathematics, July 1, 2008); Amal Amleh (Assistant Professor, Applied Mathematics, July 1, 2008).

Resignations: Anthony Bonato (July 1, 2008).

The University of Western Ontario, London, ON

Promotions: Graham Denham (Associate Professor with Tenure, July 1, 2008); Nicole Lemire (Associate Professor with Tenure, July 1, 2008).

Appointments: Matthias Franz (Assistant Professor, July 1, 2008, Toric Topology); Martin Pinsonnault (Assistant Professor, July 1, 2008, Symplectic Topology).

Awards/Distinctions: Richard Kane (Distinguished University Professor, July 1, 2008)