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★**The Riemann hypothesis.**

A resource for the afficionado and virtuoso alike.

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Expert witnesses: E. Bombieri, Problems of the millennium: the Riemann hypothesis [http://www.claymath.org/millennium/Riemann_Hypothesis/Official_Problem_Description.pdf] (94–105); Peter Sarnak, Problems of the millennium: the Riemann hypothesis (2004) [http://www.claymath.org/millennium/Riemann_Hypothesis/Sarnak_RH.pdf] (106–115); J. Brian Conrey, The Riemann hypothesis [MR1954010 (2003j:11094)] (116–129); Aleksandar Ivić, On some reasons for doubting the Riemann hypothesis [arxiv.org/abs/math/0311162] (130–160).

The experts speak for themselves: P. L. Chebyshev, Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée [J. Math. **17** (1852), 341–365] [On the function that determines all primes less than a given bound] (162–182); B. Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse [Berlin. Monatsber. Akad. **1859**, 671–680] [On the number of primes less than a given magnitude] (183–198); J. Hadamard, Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques [Bull. Soc. Math. France **24** (1896), 199–220; MR1504264] [On the distribution of the zeroes of the $\zeta(s)$ function and its arithmetical consequences] (199–221); C. de la Vallée Poussin, Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée [Belg. Mém. Courron., 8th edition, Vol. 59, No. 1, Brussels, 1899–1900; reprint, MR0079602] [On Riemann's $\zeta(s)$ function and the number of primes less than a given bound] (222–295); G. H. Hardy, Sur les zéros de la fonction $\zeta(s)$ de Riemann [C. R. Acad. Sci. Paris **158** (1914), 1012–1014] [On the zeroes of Riemann's $\zeta(s)$ function] (296–299); G. H. Hardy, Prime numbers [Brit. Ass. Rep. **85** (1915), 350–354] (300–306); G. H. Hardy and J. E. Littlewood, New proofs of the prime-number theorem and similar theorems [Quart. J. Math. **46** (1915), 215–219] (307–312); A. Weil, On the Riemann hypothesis in function-fields [MR0004242] (313–316); P. Turán, On some approximative Dirichlet-polynomials in the theory of the zeta-function of Riemann [MR0027305] (317–352); A. Selberg, An elementary proof of the prime number theorem [MR0029410] (353–362); P. Erdős, On a new method in elementary number theory which leads to an elementary proof of the prime number theorem [MR0029411] (363–374); S. Skewes, On the difference $\pi(x) - \mathrm{Li}(x)$. II [MR0067145] (375–398); C. B. Haselgrove, A disproof of a conjecture of Pólya [MR0104638] (399–404); H. Montgomery, The pair correlation of zeros of the zeta function [MR0337821] (405–418); D. J. Newman, Simple analytic proof of the prime number theorem [MR0602825] (419–423); J. Korevaar, On Newman's quick way to the prime number theorem [MR0684025] (424–432); H. Daboussi, Sur le théorème des nombres premiers [MR0741085] [On the prime number theorem] (433–437); A. Hildebrand, The prime number theorem via the large sieve [MR0859495] (438–446); D. Goldston and H. Montgomery, Pair correlation of zeros and primes in short intervals [MR1018376 (90h:11084)] (447–468); M. Agrawal, N. Kayal and N. Saxena, PRIMES is in P [MR2123939 (2006a:11170)] (469–482).

This delightfully written book on the Riemann Hypothesis is a welcome addition to the literature. It advertises itself as a resource for the aficionado and virtuoso alike, and delivers. It was a joy to read, and its structure makes it an ideal choice as a textbook for a reading course on the Riemann zeta function and its applications, especially in classes with students of diverse mathematical backgrounds and abilities.

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for all s whose real part is greater than 1. The reason this function plays such a central role in number theory is that, by unique factorization (every number can be written uniquely as a product of primes), it can also be written as

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The primes are the building blocks of number theory. Their distribution is mysterious and one of primary importance; this formula connects their properties to those of the positive integers, which are well known.

There are numerous assertions equivalent to the Riemann Hypothesis, which is typically stated as “the nontrivial zeros of the analytic continuation of $\zeta(s)$ have real part equal to $1/2$”. From the product formula, it is easy to see that $\zeta(s) \neq 0$ for the real part of s greater than 1. Let $\pi(x)$ denote the number of primes at most x. The fact that $\zeta(s)$ has no zeros with real part of s equal to 1 implies the celebrated Prime Number Theorem, namely that $\pi(x)$ is asymptotic to

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \approx \frac{x}{\log x},$$

while the Riemann Hypothesis is equivalent to “$|\pi(x) - \mathrm{Li}(x)|$ is bounded by $Cx^{1/2}\log x$ for some C”.

The book is divided into two parts. The first part, in a very entertaining and readable style, describes many of the key properties of $\zeta(s)$. There is a wonderful mix of theory and computation here, with topics such as the functional equation, zero free regions, and algorithms for computing zeros. The purpose is not to prove all these statements in full detail, but rather to summarize the key properties and relations and guide the reader through the literature. Part one concludes with two nice chapters, one on consequences of the Riemann Hypothesis and one looking at failed attempts to prove it, and then a reference list of formulas involving $\zeta(s)$ and a timeline on the Riemann Hypothesis and related subjects.

The second part is a collection of papers on the Riemann Hypothesis, and makes up about 80% of the book. It is worth repeating a quote by Eric Temple Bell given in the introduction to Chapter

12: "To appreciate the living spirit rather than the dry bones of mathematics, it is necessary to inspect the work of a master at first hand. Textbooks and treatises are an unavoidable evil. . . The very crudities of the first attack on a significant problem by a master are more illuminating than all the pretty elegance of the standard texts which has been won at the cost of perhaps centuries of finicky polishing." The first chapter contains four recent papers on the subject. The first two, by Enrico Bombieri and Peter Sarnak, were commissioned by the Clay Mathematics Institute to serve as official prize descriptions. The third is Brian Conrey's AMS Conant Prize winning article on the Riemann Hypothesis, while the fourth, by Aleksandar Ivić, describes some reasons the Riemann Hypothesis may in fact be false.

The final chapter is a collection of twenty papers by the masters: Chebyshev, Riemann, Hadamard, de la Vallée Poussin, Hardy, Hardy-Littlewood, Weil, Turán, Selberg, Erdős, Skewes, Haselgrove, Montgomery, Newman, Korevaar, Daboussi, Goldston-Montgomery, and Agrawal-Kayal-Saxena. The topics range from Chebyshev's asymptotic bounds for $\pi(x)$ to Riemann's celebrated manuscripts to the proofs of the Prime Number Theorem (both with and without complex analysis), connections to random matrix theory and applications to fast primality testing.

I thoroughly enjoyed reading this book. The expository sections in the beginning serve as a terrific introduction to the subject, and the papers by the experts are well chosen. It is a great service to have them collected in one place, and this will increase the number of mathematicians who read them.

Reviewed by *Steven Joel Miller*