SMALL PRIME SOLUTIONS OF QUADRATIC EQUATIONS

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ABSTRACT

Let $b_1, \ldots, b_5$ be non-zero integers and $n$ any integer. Suppose that $b_1 + \cdots + b_5 \equiv n \pmod{24}$ and $(b_i, b_j) = 1$ for $1 \leq i < j \leq 5$. In this paper we prove that (i) if $b_j$ are not all of the same sign, then the above quadratic equation has prime solutions satisfying $p_j \ll \sqrt{|n|} + \max(|b_j|)^{25/2+\varepsilon}$; and (ii) if all $b_j$ are positive and $n \gg \max(|b_j|)^{25+\varepsilon}$, then the quadratic equation $b_1p_1^2 + \cdots + b_5p_5^2 = n$ is soluble in primes $p_j$.

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1. INTRODUCTION

For any integer $n$, we consider quadratic equations in the form

$$b_1p_1^2 + \cdots + b_5p_5^2 = n, \quad (1.1)$$

where $p_j$ are prime variables and the coefficients $b_j$ are non-zero integers. A necessary condition for the solubility of (1.1) is

$$b_1 + \cdots + b_5 \equiv n \pmod{24}. \quad (1.2)$$

We also suppose

$$(b_i, b_j) = 1, \quad 1 \leq i < j \leq 5, \quad (1.3)$$

and write $B = \max\{2, |b_1|, \ldots, |b_5|\}$. The main results in this paper are the following two theorems.

Theorem 1. Suppose (1.2) and (1.3). If $b_1, \ldots, b_5$ are not all of the same sign, then (1.1) has solutions in primes $p_j$ satisfying

$$p_j \ll \sqrt{|n|} + B^{25/2+\varepsilon},$$

where the implied constant depends only on $\varepsilon$.

Theorem 2. Suppose (1.2) and (1.3). If $b_1, \ldots, b_5$ are all positive, then (1.1) is soluble whenever

$$n \gg B^{25+\varepsilon},$$

where the implied constant depends only on $\varepsilon$.

Theorem 2 with $b_1 = \ldots = b_5 = 1$ is a classical result of Hua [8] in 1938. The quadratic equation (1.1) in general was first studied by M.C. Liu and Tsang [16], who obtained a qualitative bound $B^A$, in the place of $B^{25/2+\varepsilon}$ and $B^{25+\varepsilon}$ in Theorems 1 and 2 above, without the explicit values of the constant $A$.

Our investigation on (1.1) is not only motivated by [8] and [16], but also by the following work on small prime solutions of the equation

$$b_1p_1 + b_2p_2 + b_3p_3 = n, \quad (1.4)$$

where $b_1, b_2, b_3, n$ are non-zero integers satisfying some necessary conditions. This problem was first raised and investigated by Baker in his well-known work [1], and was later settled qualitatively by M.C. Liu and Tsang [15]. In this problem, the constant $A$ corresponding to the 20 in our Theorem
1 is called Baker’s constant. The first author [3] proved that Baker’s constant is \( \leq 4190 \), and M.C. Liu and Wang [17] improved this to 45.

We prove our theorems by the circle method, and the idea will be explained in §2. At this stage, we only point out that in contrast to the earlier works [3][15][16][17] which treat the enlarged major arc by the Deuring-Heilbronn phenomenon, we show that in the context of this paper, the possible existence of Siegel zero does not have special influence and hence the Deuring-Heilbronn phenomenon can be avoided. This observation enables us to get better results without numerical computations.

**Notation.** As usual, \( \varphi(n) \), \( \mu(n) \), and \( \Lambda(n) \) stand for the functions of Euler, Möbius, and von Mangoldt respectively, \( d(n) \) is the divisor function. We use \( \chi \pmod{q} \) and \( \chi^0 \pmod{q} \) to denote a Dirichlet character and the principal character modulo \( q \), and \( L(s, \chi) \) is the Dirichlet \( L \)-function. \( r \sim R \) means \( R < r \leq 2R \). The letters \( c \) and \( c_j \) denote absolute positive constants, but the value of \( c \) without subscript may vary at different places. The letter \( \varepsilon \) denotes a positive constant which is arbitrarily small.

In mathematical formulae, we will write “s.t.” for “similar terms”. For example, \( "A_1B_2C_3D_4E_5+\text{s.t.}" \) means the sum of all possible terms \( A_\alpha B_\beta C_\gamma D_\delta E_\varepsilon \) with \( (\alpha, \ldots, \varepsilon) \) being any permutation of \( (1, \ldots, 5) \).

## 2. Outline of the Method

Denote by \( r(n) \) the weighted number of solutions of (1.1), i.e.

\[
r(n) = \sum_{n=b_1p_1^2 + \ldots + b_5p_5^2, M < b_jp_j^2 \leq N} (\log p_1) \cdots (\log p_5),
\]

where \( M = N/200 \). We will investigate \( r(n) \) by the circle method. To this end, we set

\[
P = (N/B)^{1/5 - \varepsilon}, \quad Q = N/(PL^{900}), \quad \text{and} \quad L = \log N.
\]

By Dirichlet’s lemma on rational approximation, each \( \alpha \in [1/Q, 1 + 1/Q] \) may be written in the form

\[
\alpha = a/q + \lambda, \quad |\lambda| \leq 1/(qQ)
\]

for some integers \( a, q \) with \( 1 \leq a \leq q \leq Q \) and \( (a, q) = 1 \). We denote by \( \mathcal{M}(a, q) \) the set of \( \alpha \) satisfying (2.2), and define the major arcs \( \mathcal{M} \) and the minor arcs \( \mathcal{m} \) as follows:

\[
\mathcal{M} = \bigcup_{q \leq P} \bigcup_{a=1}^{q} \mathcal{M}(a, q), \quad \mathcal{m} = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathcal{M}.
\]

It follows from \( 2P \leq Q \) that the major arcs \( \mathcal{M}(a, q) \) are mutually disjoint. Let

\[
S_j(\alpha) = \sum_{M < |b_jp^2| \leq N} (\log p)e(b_jp^2\alpha).
\]

Then we have

\[
r(n) = \int_0^1 S_1(\alpha) \cdots S_5(\alpha)e(-n\alpha) d\alpha = \int_{\mathcal{M}} + \int_{\mathcal{m}}.
\]
The integral on the major arcs \( \mathcal{M} \) causes the main difficulty, which is solved by the following

**Theorem 3.** Assume (1.3). Let \( \mathcal{M} \) be as in (2.3) with \( P, Q \) determined by (2.1). Then we have

\[
\int_{\mathcal{M}} S_1(\alpha) \cdots S_5(\alpha)e(-\alpha) d\alpha = \frac{1}{32} \mathcal{G}(n, P) \mathcal{I}(n) + O \left( \frac{N^{3/2}}{|b_1 \cdots b_5|^{1/2}L} \right),
\]

where \( \mathcal{G}(n, P) \) and \( \mathcal{I}(n) \) are defined in (2.6) and (2.7) respectively.

The proof of this theorem forms the bulk of this paper, \S\S 3-6. From (2.1) one sees that our major arcs are quite large. Historically, enlarged major arcs are treated by the Deuring-Heilbronn phenomenon. But here we observe that under the assumption (1.3), we can save the factor \( B^{5/2} \) in Lemma 3.1 below (in Lemma 3.8 in [16], there is an extra factor of \( B^{5/2} \) on the right-hand side). With this saving, (2.5) can be derived from the large sieve inequality, Gallagher’s lemma and classical results on the distribution of zeros of \( L \)-functions. This approach has also been used by Bauer, M.C. Liu, and Zhan [2], and by M.C. Liu, Zhan, and the second author [12][13].

To derive Theorems 1 and 2 from Theorem 3, we need to bound \( \mathcal{G}(n, P) \) and \( \mathcal{I}(n) \) from below. For \( \chi \mod q \), we define

\[
C(\chi, a) = \sum_{h=1}^{q} \bar{\chi}(h) e \left( \frac{ah^2}{q} \right), \quad C(q, a) = C(\chi^0, a).
\]

If \( \chi_1, \ldots, \chi_5 \) are characters \( \mod q \), then we write

\[
B(n, q, \chi_1, \ldots, \chi_5) = \sum_{h=1}^{q} \sum_{(b_1, q)=1} e \left( -\frac{hn}{q} \right) C(\chi_1, b_1h) \cdots C(\chi_5, b_5h),
\]

\[
B(n, q) = B(n, q, \chi_1^0, \ldots, \chi_5^0), \quad A(n, q) = \frac{B(n, q)}{\varphi(q)}, \quad \mathcal{G}(n, x) = \sum_{q \leq x} A(n, q). \tag{2.6}
\]

**Lemma 2.1.** Assuming (1.2), we have \( \mathcal{G}(n, P) \gg (\log \log B)^{-c_1} \) for some constant \( c_1 > 0 \).

*Proof.* This is Lemma 2.1 in [4]. \( \square \)

**Lemma 2.2.** Suppose (1.3) and

(i) \( b_j \)'s are not all of the same sign and \( N \geq 10|n| \); or
(ii) all \( b_j \)'s are positive and \( n = N \).

Then we have

\[
\mathcal{I}(n) := \sum_{b_1m_1 + \ldots + b_5m_5 = n \atop M < |b_j|m_j \leq N} (m_1 \cdots m_5)^{-1/2} \times \frac{N^{3/2}}{|b_1 \cdots b_5|^{1/2}}, \tag{2.7}
\]

*Proof.* This is Lemma 2.2 in [4]. \( \square \)

We now derive Theorems 1 and 2 from Theorem 3 and Lemmas 2.1 and 2.2.

**Proofs of Theorems 1 and 2.** We start from (2.4) and let \( N_j = N/|b_j| \). To estimate the integral on \( \mathcal{M} \), one appeals to the estimate on p.151 in [16]:

\[
S_5(\alpha) \ll N_5^{1/2+\varepsilon}(|b_5|P^{-1} + N_5^{-1/4} + QN_5^{-1/4})^2 \ll N_5^{1/2+\varepsilon}(|b_5|/P)^{1/4} \ll N_5^{1/2-\varepsilon}(|b_5|P)^{-1/4}. \tag{2.8}
\]
Also, we have the following mean-value estimate for $S_j(\alpha)$:

$$\int_0^1 |S_j(\alpha)|^4 \, d\alpha \ll L^4 \sum_{\substack{m_1^2 + m_2^2 = m_3^2 + m_4^2 \\ m_5^2 \leq N_j, \, \nu = 1, \ldots, 4}} 1 \ll N_j^{1+\varepsilon},$$

which in combination with Hölder’s inequality gives

$$\int_0^1 |S_1(\alpha) \cdots S_4(\alpha)| \, d\alpha \ll \frac{N^{1+\varepsilon}}{|b_1 \cdots b_4|^{1/4}}.$$  \hspace{1cm} (2.9)

It therefore follows from (2.8) and (2.9) that

$$\left| \int_m \right| \ll \frac{N^{3/2+\varepsilon}}{|b_1 \cdots b_5|^{1/4} P^{1/4}}.$$  \hspace{1cm} (2.10)

The contribution from the major arcs can be handled by Theorem 3, which together with (2.10) gives

$$r(n) = \frac{1}{32} \mathcal{S}(n, P)3(n) + O\left( \frac{N^{3/2}}{|b_1 \cdots b_5|^{1/2} L} + \frac{N^{3/2+\varepsilon}}{|b_1 \cdots b_5|^{1/4} P^{1/4}} \right).$$

Now assume the conditions (i) or (ii) in Lemma 2.2. Applying Lemmas 2.1 and 2.2 to the above formula, we conclude

$$r(n) \gg |b_1 \cdots b_5|^{-1/2} N^{3/2} (\log \log B)^{-c_1}$$

provided that $P \gg N^\varepsilon |b_1 \cdots b_5|$, or equivalently $N \gg B^{1+\varepsilon}|b_1 \cdots b_5|^8$. This proves Theorems 1 and 2.

3. AN EXPLICIT EXPRESSION

In this section, we establish an explicit expression for the integral in Theorem 3 (see Lemma 3.5 below), from which and the estimates in §§4-6 we can derive Theorem 3 at the end of §6.

**Lemma 3.1.** Let $\chi_j \mod r_j$ with $j = 1, \ldots, 5$ be primitive characters, $r_0 = [r_1, \ldots, r_5]$, and $\chi^0$ the principal character $\mod q$. Then

$$\sum_{q \leq \varphi^5(q)} \frac{1}{\varphi^5(q)} |B(n,q;\chi_1\chi^0, \ldots, \chi_5\chi^0)| \ll r_0^{-1+\varepsilon} \log^{215} x.$$  \hspace{1cm} ($\varphi(q)$ denotes the Euler totient function.)

**Proof.** This is Lemma 3.1 in [4].\hfill $\square$

For $j = 1, \ldots, 5$, recall $N_j = N/|b_j|$, and set

$$M_j = M/|b_j|, \quad V_j(\lambda) = \sum_{M < |b_j| m^2 \leq N} e(b_j m^2 \lambda),$$

and

$$W_j(\chi, \lambda) = \sum_{M < |b_j| p^2 \leq N} \log p) \chi(p) e(b_j p^2 \lambda) - \delta_{\chi} \sum_{M < |b_j| m^2 \leq N} e(b_j m^2 \lambda),$$  \hspace{1cm} (3.1)

where $\delta_{\chi} = 1$ or 0 according as $\chi$ is principal or not. Also, for $g \geq 1$, we define

$$J_j(g) = \sum_{r \leq P} [g, r]^{-1+\varepsilon} \sum_{\chi \mod r} \max_{|\lambda| \leq 1/(rQ)} |W_j(\chi, \lambda)|,$$
and
\[ K_j(g) = \sum_{r \leq P} [g, r]^{-1+\varepsilon} \sum_{\chi \mod r} * \left( \int_{-1/(rQ)}^{1/(rQ)} |W_j(\chi, \lambda)|^2 d\lambda \right)^{1/2}, \]

where \( \sum_{\chi \mod r} * \) is over all the primitive characters modulo \( r \) and \([g, r]\) is the least common multiple of \( g \) and \( r \).

Our Theorem 1.2 depends on the following three lemmas, which will be proved in §§4-6.

**Lemma 3.2.** For \( P, Q \) satisfying (2.1), we have
\[ J_j(g) \ll g^{-1+\varepsilon} N_j^{1/2} L^c. \]

**Lemma 3.3.** Let \( P, Q \) be as in (2.1). For \( g = 1 \) Lemma 3.2 can be improved to
\[ J_j(1) \ll N_j^{1/2} L^{-A}, \]
where \( A > 0 \) is arbitrary.

**Lemma 3.4.** For \( P, Q \) as in (2.1), we have
\[ K_j(g) \ll g^{-1+\varepsilon} L^c. \]

Now we state the main result of this section.

**Lemma 3.5.** Let \( \mathcal{M} \) be as in (2.3). Then
\[
\int_{\mathcal{M}} S_1(\alpha) \cdots S_5(\alpha) e(-n\alpha) d\alpha - \frac{1}{32} \mathcal{S}(n, P) \mathcal{J}(n) \\
\ll (J_1 J_2 J_3 K_4 K_5) L^{c_2} + (J_1 J_2 J_3 |b_5|^{-1/2} + \text{s.t.}) L^{c_2} + (J_1 J_2 J_3 |b_4|^{-1/2} |b_5|^{-1/2} + \text{s.t.}) L^{c_2} \\
+ (J_1 J_2 N_3^{1/2} |b_4|^{-1/2} |b_5|^{-1/2} + \text{s.t.}) L^{c_2} + (J_1 N_2^{1/2} N_3^{1/2} |b_4|^{-1/2} |b_5|^{-1/2} + \text{s.t.}) L^{c_2} \\
+ |b_1 \cdots b_5|^{-1/2} N^{3/2} L^{-1},
\]
where \( c_2 = 2^{15} + 1 \) and “s.t.” means similar terms as explained at the end of §1.

**Proof.** This is Lemma 3.2 in [4].

4. Estimation of \( J \) for General \( g \)

We have
\[ J_j(g) \ll L \max_{R \leq P} J_j(g, R) \]
where \( J_j(g, R) \) is defined similarly to \( J_j(g) \) except that the sum is over \( r \sim R \). The estimation of \( J_j(R) \) falls naturally into two cases according as \( R \) is small or large. For \( R > L^C \), where \( C \) is some positive constant, one appeals to contour integration, mean-value estimates for the Dirichlet \( L \)-functions or their derivatives, the large sieve inequality, and Heath-Brown’s identity. While for \( R \leq L^C \), one uses the classical zero-density estimates and zero-free region for the Dirichlet \( L \)-functions.

We first establish the following result for large \( R \). In Lemma ?? we shall consider small \( R \).
**Lemma 4.1.** There exists a constant $c > 0$ such that $w$

$$J_j(R) \ll g^{-1+\varepsilon} N_j^{1/2} L^c.$$  

To prove this result, it suffices to show that

$$\sum_{r \sim R} [g, r]^{-1+\varepsilon} \sum_{\chi \mod r} * \max_{|\lambda| \leq 1/(rQ)} |W_j(\chi, \lambda)| \ll g^{-1+\varepsilon} N_j^{1/2} L^c. \quad (4.1)$$

Let

$$\tilde{W}_j(\chi, \lambda) = \sum_{M < |b_j|m^2 \leq N} (\Lambda(m) \chi(m) - \delta_\chi) e(b_j m^2 \lambda). \quad (4.2)$$

Then

$$W_j(\chi, \lambda) - \tilde{W}_j(\chi, \lambda) = - \sum_{m \geq 2} \sum_{M < |b_j|p^{2n} \leq N} (\log p) \chi(p) e(b_j p^{2n} \lambda) \ll N_j^{1/4}. \quad (4.3)$$

This contributes to (4.1) in amount

$$\ll N_j^{1/4} \sum_{r \leq P} [g, r]^{-1+\varepsilon} r \leq g^{-1+\varepsilon} N_j^{1/4} \sum_d d \sum_{r \leq P/d} r^\varepsilon$$

$$\ll g^{-1+\varepsilon} N_j^{1/4} P^{1+2\varepsilon} \ll g^{-1+\varepsilon} N_j^{1/2}$$

where we have used $[g, r](g, r) = g^r$. Thus (4.1) is a consequence of the estimate

$$\sum_{r \sim R} [g, r]^{-1+\varepsilon} \sum_{\chi \mod r} * \max_{|\lambda| \leq 1/(rQ)} |\tilde{W}_j(\chi, \lambda)| \ll g^{-1+\varepsilon} N_j^{1/2} L^c. \quad (4.4)$$

Let $M_j^{1/2} < u \leq N_j^{1/2}$, and let $D_1, \ldots, D_{10}$ be positive numbers such that

$$2^{-10} M_j^{1/2} \leq D_1 \cdots D_{10} < u, \quad \text{and} \quad 2D_1, \ldots, 2D_{10} \leq u^{1/5}.$$  

For $\nu = 1, \ldots, 10$ let

$$a_\nu(m) = \begin{cases} 
\log m & \text{if } \nu = 1; \\
1 & \text{if } \nu = 2, 3, 4, 5; \\
\mu(m) & \text{if } \nu = 6, 7, 8, 9, 10.
\end{cases}$$

We define the following functions of a complex variable $s$:

$$f_\nu(s) = f_\nu(s, \chi) = \sum_{m \sim D_\nu} \frac{a_\nu(m) \chi(m)}{m^s}, \quad F(s) = F(s, \chi) = f_1(s) \cdots f_{10}(s).$$

Now we recall Heath-Brown’s identity (see Lemma 1 in [7]) for $k = 5$, which states that

$$\frac{\zeta'}{\zeta}(s) = \sum_{\nu=1}^{5} \frac{\zeta'}{\zeta}(s)(-1)^{\nu-1} \zeta^{\nu-1}(s) G^\nu(s) + \frac{\zeta'}{\zeta}(s)(1 - \zeta(s) G(s))^5,$$

where $\zeta(s)$ is the Riemann zeta-function, and $G(s) = \sum_{m \leq u^{1/5}} \mu(m) m^{-s}$. The reason why we choose $k = 5$ is that the identity with $k \leq 4$ will give weaker results, and when $k \geq 6$ it produces the same estimate as the case $k = 5$. Equating coefficients of the Dirichlet series on both sides provides an
identity for $-\Lambda(m)$. Also, for $m \leq u$ the coefficient of $m^{-s}$ in $-(\zeta'/\zeta)(s)(1 - \zeta(s)G(s))^5$ is zero. Thus,

$$
\Lambda(m) = \sum_{\nu=1}^{5} \binom{5}{\nu} (-1)^{\nu-1} \sum_{\substack{m_1 \cdots m_{2\nu} = m \\ m_{\nu+1} \cdots m_{2\nu} \leq u}} (\log m_1) \mu(m_{\nu+1}) \cdots \mu(m_{2\nu}).
$$

Applying this identity to the sum

$$
\sum_{M_j^{1/2} < m \leq u} \Lambda(m) \chi(m), \quad (4.5)
$$

one finds that (4.5) is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$
\sigma(u; D) = \sum_{m_1 \sim D_1} \cdots \sum_{m_{10} \sim D_{10}} a_1(m_1) \chi(m_1) \cdots a_{10}(m_{10}) \chi(m_{10})
$$

where $D$ denotes the vector $(D_1, \ldots, D_{10})$. By using Perron’s summation formula (see for example, Lemma 3.12 in [19]) and then shifting the contour to the left, the above $\sigma(u; D)$ is

$$
= \frac{1}{2\pi i} \int_{1+1/L-iT}^{1+1/L+iT} F(s, \chi) \frac{u^s - M_j^{3/2}}{s} ds + O \left( \frac{N_j^{1/2} L^2}{T} \right)
$$

$$
= \frac{1}{2\pi i} \left\{ \int_{1+1/L-iT}^{1/2-iT} + \int_{1/2-iT}^{1/2+iT} + \int_{1/2+iT}^{1+1/L+iT} \right\} + O \left( \frac{N_j^{1/2} L^2}{T} \right),
$$

where $T$ is a parameter satisfying $2 \leq T \leq N_j^{1/2}$. The integral on the two horizontal segments above can be easily estimated as

$$
\ll \max_{1/2 \leq \sigma \leq 1+1/L} |F(\sigma \pm iT, \chi)| \frac{u^\sigma}{T} \ll \max_{1/2 \leq \sigma \leq 1+1/L} N_j^{(1-\sigma)/2} L \frac{u^\sigma}{T} \ll \frac{N_j^{1/2} L}{T}
$$

on using the trivial estimate

$$
F(\sigma \pm iT, \chi) \ll |f_1(\sigma \pm iT, \chi)| \cdots |f_{10}(\sigma \pm iT, \chi)| \ll (D_1^{1-\sigma} L)D_2^{1-\sigma} \cdots D_{10}^{1-\sigma} \ll N_j^{(1-\sigma)/2} L.
$$

Thus,

$$
\sigma(u; D) = \frac{1}{2\pi} \int_{-T}^{T} F \left( \frac{1}{2} + it, \chi \right) \frac{u^{1/2 + it} - M_j^{1/2 + it}}{\frac{1}{2} + it} dt + O \left( \frac{N_j^{1/2} L^2}{T} \right).
$$

Since $R > L^C$ (so $\chi \neq \chi^0$), we have in (4.2) that

$$
\hat{W}_j(\chi, \lambda) = \sum_{M < |b_j| m^2 \leq N} \Lambda(m) \chi(m) e(b_j m^2 \lambda) = \int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda) d \left\{ \sum_{M_j^{1/2} < m \leq u} \Lambda(m) \chi(m) \right\},
$$

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and consequently \( \tilde{W}(\chi, \lambda) \) is a linear combination \( O(L^{10}) \) terms, each of which is of the form

\[
\int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda) d\sigma(u; D) = \frac{1}{2\pi} \int_{-T}^{T} F \left( \frac{1}{2} + it, \chi \right) \int_{M_j^{1/2}}^{N_j^{1/2}} u^{-1/2+\epsilon} e(b_j u^2 \lambda) du \, dt + O \left( \frac{N_j^{1/2} L^2}{T} (1 + |\lambda|N) \right).
\]

By taking \( T = N_j^{1/2} \) and changing variables in the inner integral, we deduce from the above formulae that

\[
|\tilde{W}_j(\chi, \lambda)| \ll L^{10} \max_D \left| \int_{-T}^{T} F \left( \frac{1}{2} + it, \chi \right) \int_{M_j}^{N_j} v^{-3/4} e \left( \frac{t}{4\pi} \log v + b_j \lambda v \right) dv \, dt \right| + PL^{9012}, \tag{4.6}
\]

where the maximum is taken over all \( D = (D_1, \ldots, D_{30}) \). Since

\[
\frac{d}{dv} \left( \frac{t}{4\pi} \log v + b_j \lambda v \right) = \frac{t}{4\pi v} + b_j \lambda, \quad \frac{d^2}{dv^2} \left( \frac{t}{4\pi} \log v + b_j \lambda v \right) = -\frac{t}{4\pi v^2},
\]

by Lemmas 4.4 and 4.3 in [19], the inner integral in (4.6) can be estimated as

\[
\ll N_j^{-3/4} \min \left\{ \frac{N_j}{(|t| + 1)^{1/2}}, \min_{M_j < v \leq N_j} \frac{N_j}{|t + 4\pi b_j \lambda v|} \right\} \ll \begin{cases} N_j^{1/4} (|t| + 1)^{-1/2} & \text{if } |t| \leq T_0; \\ N_j^{1/4} |t|^{-1} & \text{if } T_0 < |t| \leq T; \end{cases} \tag{4.7}
\]

where \( T_0 = 8\pi N/(RQ) \). Here the choice of \( T_0 \) is to ensure that \(|t + 4\pi b_j \lambda v| > |t|/2\) whenever \(|t| > T_0\); in fact,

\[
|t + 4\pi b_j \lambda v| \geq |t| - 4\pi |b_j v|/(RQ) > |t|/2 + T_0/2 - 4\pi N/(RQ) = |t|/2.
\]

It therefore follows from (4.6) and (4.7) that the lemma (more precisely, (4.4)) is a consequence of the following two estimates: For \( 0 < T_1 \leq T_0 \), we have

\[
\sum_{r \sim R} [g, r]^{-1+\epsilon} \sum_{\chi \bmod r} \int_{T_1}^{2T_1} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \ll g^{-1+\epsilon} N_j^{1/4} (T_1 + 1)^{1/2} L^c, \tag{4.8}
\]

while for \( T_0 < T_2 \leq T \), we have

\[
\sum_{r \sim R} [g, r]^{-1+\epsilon} \sum_{\chi \bmod r} \int_{T_2}^{2T_2} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \ll g^{-1+\epsilon} N_j^{1/4} T_2 L^c. \tag{4.9}
\]

Both (4.8) and (4.9) are deduced from the following bound, which is Lemma 5.2 in [12].

**Lemma 4.2.** Let \( F(s, \chi) \) be defined as above. Then for any \( 1 \leq R \leq u^2 \) and \( T > 0 \),

\[
\sum_{r \sim Rd} \sum_{\chi \bmod r} \left| \int_{T}^{2T} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \right| \ll \left( \frac{R^2}{d} T + \frac{R}{d^{1/2}} T^{1/2} u^{3/10} + u^{1/2} \right) \log^c u.
\]

**Proof.** This is Theorem 1.1 in [11]. \( \square \)

Now we can complete the proof of Lemma 4.1 immediately.
Proof of Lemma 4.1. By taking $T_3 = T_1$ in Lemma 4.2, the left-hand side of (4.8) is now
\[
\ll g^{-1+\varepsilon} \sum_{\substack{d \mid g \\ d \leq R}} \left( \frac{R}{d} \right)^{-1+\varepsilon} \sum_{r \sim R/d} \sum_{\chi \mod r} \int_{T_1}^{2T_1} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt
\]
By Lemma 4.2, the above quantity can be estimated as
\[
\ll g^{-1+\varepsilon} \tau(g) \left( \frac{R^2}{d} T_1 + \frac{R}{d} T_1^{1/2} N_j^{3/20} + N_j^{1/4} \right) L^c
\]
\[
\ll g^{-1+\varepsilon} N_j^{1/2} (T_1 + 1)^{1/2} L^c,
\]
provided that $R \leq P = (N/B)^{1/5-\varepsilon}$. This establishes (4.8). Similarly we can prove (4.9) by taking $T_3 = T_2$ in Lemma 4.2. Lemma 4.1 now follows. □

5. Estimation of $J$ for $g = 1$

In this section we give the proof of Lemma 2.3.

Proof of Lemma 2.3. Lemma 2.3 is the same as that of Lemma 2.2 except for the saving $L^{-A}$ on the right-hand side. Because of this saving, we have to distinguish two cases according as $R$ small or large.

Consider firstly the case $L^C < R \leq P$ where $C$ is a constant depending on $A$. Recall now $g = 1$. We follow the proof of Lemma 2.2 until (4.11) and (4.12), and find that Lemma 2.3 is a consequence of the following two estimates: For $0 < T_1 \leq T_0$, we have
\[
\sum_{r \sim R} r^{-1+\varepsilon} \sum_{\chi \mod r} \int_{T_1}^{2T_1} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \ll N_j^{1/4} (T_1 + 1)^{1/2} L^{-A},
\]
while for $T_0 < T_2 \leq T$, we have
\[
\sum_{r \sim R} r^{-1+\varepsilon} \sum_{\chi \mod r} \int_{T_2}^{2T_2} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \ll N_j^{1/4} T_2 L^{-A}.
\]
Taking $g = 1$ in (4.13), we see that the left-hand side of (5.1) is
\[
\ll (R^{1+\varepsilon} T_1 + R^{\varepsilon} T_1^{1/2} N_j^{3/20} + R^{-1+\varepsilon} N_j^{1/4}) L^c
\]
\[
\ll N_j^{1/4} (T_1 + 1)^{1/2} L^{-A},
\]
if $L^C < R \leq P = N^{1/5-\varepsilon}$ with a sufficiently large $C$. Here $L^C < R$ guarantees that the term $R^{-1+\varepsilon} N_j^{1/4} L^c$ is dominated by $N_j^{1/4} (T_1 + 1)^{1/2} L^{-A}$. This establishes (5.1). Similarly we can prove (5.2) by taking $T = T_2$ in Lemma 4.1. This proves Lemma 2.3 in the first case.

Now we turn to the second case $R \leq L^B$.

We use the explicit formula (see [5], p.109 and 120)
\[
\sum_{m \leq u} \Lambda(m) \chi(m) = \delta \chi u - \sum_{|\gamma| \leq T} \frac{w^\gamma}{\rho} + O \left\{ \left( \frac{u}{T} + 1 \right) \log^2 (quT) \right\}
\]
where $\rho = \beta + i\gamma$ is a non-trivial zero of the function $L(s, \chi)$, and $2 \leq T \leq u$ is a parameter. Taking $T = N_j^{1/5}$ in (5.3), and then inserting it into $\tilde{W}_j(\chi, \lambda)$, we get by $M_j^{1/2} < u \leq N_j^{1/2}$, $M_j = N_j/200$, and (4.2) that

$$\tilde{W}_j(\chi, \lambda) = \int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda) d\left\{ \sum_{n \leq u} (\Lambda(m) \chi(m) - \delta_\chi) \right\}$$

$$= - \int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda) \sum_{|\gamma| \leq N_j^{1/6}} u^{\sigma-1} du + O(N_j^{1/3} (1 + |\lambda N|) L^2)$$

$$\ll N_j^{1/2} \sum_{|\gamma| \leq N_j^{1/6}} N_j^{(\beta-1)/2} + O(N_j^{1/2} PT^{-1} L^{9002}).$$

Now we need Satz VIII.6.2 in Prachar [18], which states that $\prod_{\chi \mod q} L(s, \chi)$ is zero-free in the region $\sigma \geq 1 - \eta(T), |t| \leq T$ except for the possible Siegel zero, where $\eta(T) = c_3 \log^{-4/5} T$. But by Siegel’s theorem (see for example [5], §21) the Siegel zero does not exist in the present situation, since $r \leq L^C$. We also need the zero-density estimate (see e.g. Huxley [10]):

$$N^*(\alpha, q, T) \ll (qT)^{12(1-\alpha)/5} \log^c (qT),$$

where $N^*(\alpha, q, T)$ denotes the number of zeros of $\prod_{\chi \mod q} L(s, \chi)$ in the region $\text{Re} s \geq \alpha, |\text{Im} s| \leq T$. Thus,

$$\sum_{|\gamma| \leq N_j^{1/6}} N_j^{(\beta-1)/2} \ll L^c \int_0^{1-\eta(N_j^{1/5})} (N_j^{1/5})^{12(1-\alpha)/5} N_j^{(\alpha-1)/2} d\alpha \ll L^c N_j^{-\eta(N_j^{1/5})/50} \ll \exp(-c_4 L^{1/5}).$$

Consequently,

$$\sum_{r \sim R} r^{-1+\varepsilon} \sum_{\chi \mod r} \max_{|\lambda| \leq 1/(rQ)} |\tilde{W}_j(\chi, \lambda)| \ll N_j^{1/2} L^{-A}, \quad (5.4)$$

where $R \leq L^C$, and $A > 0$ is arbitrary. Lemma ?? now follows from (5.4) and (4.3).

6. Estimation of $K$

In this section, we give the proof of Lemma 2.4.

In (2.6), we replace $W_j(\chi, \lambda)$ by $\tilde{W}_j(\chi, \lambda)$. By (4.5) and (1.3), the resulting error is

$$\ll \sum_{r \leq P} [g, r]^{-1+\varepsilon} r^{1/2} N_j^{1/4} Q^{1/2} \ll \frac{N_j^{1/4}}{Q^{1/2}} \sum_{r \leq P} [g, r]^{-1+\varepsilon} r^{1/2}$$

$$\ll g^{-1+\varepsilon} N_j^{1/4} Q^{1/2} \sum_{d|g} d^{-1-\varepsilon} \sum_{r \leq P} r^{-1/2+\varepsilon} \ll g^{-1+\varepsilon} \tau(g) \frac{N_j^{1/4} P^{1/2+\varepsilon}}{Q^{1/2}}$$

$$\ll g^{-1+2\varepsilon} |b_j|^{-1/2} L^c.$$
Thus to establish (??), it suffices to show that

$$
\sum_{r - R} [g, r]^{-1 + \varepsilon} \sum_{\chi \bmod r} * \left( \int_{-1/(rQ)}^{1/(rQ)} |\hat{W}_j(\chi, \lambda)|^2d\lambda \right)^{1/2} \ll |b_j|^{-1/2}g^{-1 + \varepsilon} L^c
$$

holds for $R \leq P$ and some $c > 0$.

By Gallagher’s lemma (see [6], Lemma 1), we have

$$
\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}_j(\chi, \lambda)|^2d\lambda \ll \left( \frac{1}{rQ} \right)^2 \int_{-\infty}^{\infty} \left| \sum_{v < |b_j|} \frac{(\Lambda(m)\chi(m) - \delta_\chi)}{v + rQ} \right|^2 dv
$$

$$
\ll \left( \frac{1}{rQ} \right)^2 \int_{M - rQ}^N \sum_{v < |b_j|} \left| \frac{(\Lambda(m)\chi(m) - \delta_\chi)}{v + rQ} \right|^2 dv. \tag{6.2}
$$

Let $X = \max(v, M)/|b_j|$ and $Y = \min(v + rQ, N)/|b_j|$. Then the sum in (6.2) can be written as

$$
\sum_{X < m^2 \leq Y} (\Lambda(m)\chi(m) - \delta_\chi). \tag{6.3}
$$

Before estimating (6.3), we observe first that, for any $0 < \beta < 1$,

$$
Y^\beta - X^\beta \ll \frac{(v + rQ)^\beta - v^\beta}{|b_j|^\beta} = \frac{v^\beta((1 + rQ/v)^\beta - 1)}{|b_j|^\beta} \ll \frac{rQ}{|b_j|^\beta M^{1 - \beta}}, \tag{6.4}
$$

where in the last step we have used $M - rQ \leq v \leq N$ and $rQ \leq 2rQ \leq 2PQ \ll ML^{-9000}$.

In the case $\chi = \chi^0 \bmod 1$, the quantity in (6.3) is

$$
\ll Y^{1/2} - X^{1/2} \ll |b_j|^{-1/2}M^{-1/2}Q
$$

by (6.4) with $r = 1$. This contributes to (6.2) acceptably.

For other $\chi$, we have $\delta_\chi = 0$ in (6.3). Using Heath-Brown’s identity to this sum, and applying Perron’s formula as before, we see that (6.3) is a linear combination of $O(L^{10})$ terms, each of which has the form

$$
\frac{1}{2\pi} \int_{-T}^{T} F \left( \frac{1}{2} + it, \chi \right) \frac{Y^{\frac{1}{2}(i/2 + it)} - X^{\frac{1}{2}(i/2 + it)}}{i/2 + it} dt + O \left( \frac{N_j^{1/2}L^2}{T} \right),
$$

where $D, F(s, \chi)$ are as in §4, and $T$ is a parameter satisfying $2 \leq T \leq N_j^{1/2}$. One easily sees that

$$
\frac{Y^{\frac{1}{2}(i/2 + it)} - X^{\frac{1}{2}(i/2 + it)}}{i/2 + it} = \frac{1}{2} \int_{X}^{Y} u^{-3/4 + it/2} du = \frac{1}{2} \int_{X}^{Y} u^{-3/4} e \left( \frac{t}{4\pi} \log u \right) du.
$$

The integral can be easily estimated by (6.4) as $\ll Y^{1/4} - X^{1/4} \ll |b_j|^{-1/4}M^{-3/4}RQ$. On the other hand, one has trivially

$$
\frac{Y^{\frac{1}{2}(i/2 + it)} - X^{\frac{1}{2}(i/2 + it)}}{i/2 + it} \ll \frac{Y^{1/4}}{|t|} \ll \frac{N_j^{1/4}}{|t|}.
$$
Collecting the two upper bounds, we get
\[ \frac{Y^{1/(2 + \mu t)} - X^{1/(2 + \mu t)}}{2 + \mu t} \ll \min \left( \frac{RQ}{M^{3/4} |b_j|^{1/4}}, \frac{N_j^{1/4}}{|t|} \right) \ll \frac{1}{|b_j|^{1/4}} \min \left( \frac{RQ}{N^{3/4}}, \frac{N^{1/4}}{|t|} \right). \]

Taking \( T = N_j^{1/2} \) and \( T_0 = N/(QR) \), we see that
\[ \sigma(u; D) \ll \frac{RQ}{|b_j|^{1/4} N^{3/4}} \int_{|t| \leq T_0} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt + \frac{N^{1/4}}{|b_j|^{1/4}} \int_{T_0 < |t| \leq T} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt + O(L^2). \]

And consequently (6.2) becomes
\[ \int_{-1/(rQ)}^{1/(rQ)} |\tilde{W}(\chi, \lambda)|^2 d\lambda \ll \frac{L^{20}}{|b_j|^{1/2} N^{1/2}} \max_D \left( \int_{|t| \leq T_0} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \right)^2 + \frac{N^{3/2} L^{20}}{|b_j|^{1/2} (QR)^2} \max_D \left( \int_{T_0 < |t| \leq T} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \right)^2 + \frac{NL^{24}}{(QR)^2}. \]

Now the left-hand side of (6.1) is
\[ \ll \frac{L^{10}}{|b_j|^{1/4} N^{1/4}} \max_D \sum_{r \sim R \chi \mod \rho} \sum_{r \sim R \chi \mod \rho} \int_{|t| \leq T_0} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt + \frac{N^{3/4} L^{10}}{|b_j|^{1/4} RQ} \max_D \sum_{r \sim R \chi \mod \rho} \sum_{r \sim R \chi \mod \rho} \int_{T_0 < |t| \leq T} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt + \frac{N^{1/2} g^{-1+\varepsilon} L^{12}}{Q}. \]

Thus, to prove (6.1) it suffices to show that the estimate
\[ \sum_{r \sim R} \frac{1}{g^{1+\varepsilon}} \sum_{\chi \mod \rho} \int_{T_1}^{2T_1} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \ll g^{-1+\varepsilon} N_j^{1/4} L^5 \]  

holds for \( R \leq P \) and \( 0 < T_1 \leq T_0 \), and
\[ \sum_{r \sim R} \frac{1}{g^{1+\varepsilon}} \sum_{\chi \mod \rho} \int_{T_2}^{2T_2} \left| F \left( \frac{1}{2} + it, \chi \right) \right| dt \ll g^{-1+\varepsilon} \left( \frac{RQ}{|b_j|^{1/4} N^{3/4}} \right) T_2 L^5 \]  

holds for \( R \leq P \) and \( T_0 < T_2 \leq T \).

The estimates (6.5) and (6.6) follows from Lemma 4.2. The proof of Lemma ?? is complete.

**Proof of Theorem 3.** Collecting Lemmas 3.5, 4.1, ?? and ??, we get Theorem 3. \( \square \)

**REFERENCES**


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