A Local Riemann Hypothesis, I

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Preliminary Version

In Tate's thesis [30], Hecke L-functions are studied by means of the local integrals

$$\zeta(s, \nu, f) = \int_{F} f(x) \nu(x) \left| x \right|^s \, dx,$$

where $f$ is an element of the Schwartz space $S(F)$ on a local field $F$, and $\nu$ is a character of $F^\times$. Weil [35] defined a representation $\omega = \omega_{\psi}$ of the metaplectic group $\tilde{SL}(2, F)$ on $S(F)$. We consider the restriction of $\omega$ to the special orthogonal group $SO(2)$ of $\tilde{SL}(2, F)$, corresponding to the quadratic form $x^2 + y^2$. If $-1$ is not a square in $F$, this representation is multiplicity free, and $S(F)$ decomposes into a direct sum of one-dimensional invariant subspaces. The Local Riemann Hypothesis is the assertion that if $f$ lies in one of these spaces, then the zeros of the local integral $\zeta(s, \nu, f)$ lie on the line $\text{Re}(s) = \frac{1}{2}$. (We refer to the text for the correct statement if $-1$ is a square.) This is proved in a substantial number of cases, in this paper and its companion piece by Kurlberg [19].

If $F = \mathbb{R}$, we will prove an extension of this result to the harmonic oscillator in $n$-dimensions. This result may be formulated in a way that makes sense over a $p$-adic field, though we have not investigated this yet. In this connection, we also have a reciprocity law for the values at negative integers of the Laguerre polynomials, and a geometrical interpretation of these values.

We will also state a certain conjecture, that if the spherical Whittaker function of a spherical representation of $GL(n, \mathbb{R})$ which is a functorial lift from $GL(2, \mathbb{R})$ vanishes anywhere on the group, then the representation is tempered. This generalizes a theorem of Pólya on the zeros of Bessel functions.

Up-to-date information on this topic may be found on the world-wide-web at:

http://match.stanford.edu/rh/

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1. The zeros of the Mellin transforms of Hermite polynomials. For the quantum mechanical harmonic oscillator see Weyl [36], and Cartier [7].

We recall the result of Bump and Ng [5], showing that the Mellin transforms of the Hermite functions have their zeros on the line \( \text{Re}(s) = \frac{1}{2} \). (At first Bump and Ng considered the case of \( H_n \) with \( n \) even, and Vaaler pointed out that the case \( n \) odd could be added.)

Our normalizations will be different than in [5]. Let

\[
f_n(x) = 2^{-n/2} M_n(\sqrt{2\pi} x) e^{-x^2},
\]

where the Hermite polynomials are defined by

\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]

The \( f_n \) are the eigenfunctions of the Hamiltonian \( x^2 - \frac{1}{4\pi^2} \frac{d^2}{dx^2} \) of the quantum mechanical harmonic oscillator. That is, they satisfy the Schrödinger equation

\[
\left(x^2 - \frac{1}{4\pi^2} \frac{d^2}{dx^2}\right) f_n = \frac{2n + 1}{2\pi} f_n.
\]

Define polynomials \( p_n \) by

\[
M_n(s) = \begin{cases} 
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) p_n(s) & \text{if } n \text{ is even;} \\
\frac{n}{s-(n+1)/2} \Gamma\left(\frac{s+(n+1)}{2}\right) \sqrt{2n} p_n(s) & \text{if } n \text{ is odd.}
\end{cases}
\]

where the Mellin transform

\[
M_n(s) = \int_0^\infty f_n(x) x^s \frac{dx}{x}.
\]

We have

\[
f_{n+1}(x) = \left(\sqrt{2\pi} x - \frac{1}{\sqrt{2\pi}} \frac{d}{dx}\right) f_n(x),
\]

and consequently, integrating by parts, we have

\[
M_{n+1}(s) = \sqrt{2n} M_n(s + 1) + \frac{s-1}{\sqrt{2\pi}} M_n(s-1).
\]

This implies that

\[
p_{n+1}(s) = \begin{cases} 
p_n(s + 1) + p_n(s - 1) & \text{if } n \text{ is even;} \\
sp_n(s+1) + (s-1)p_n(s-1) & \text{if } n \text{ is odd.}
\end{cases}
\]
The polynomials \( p_n \) have certain properties in common with the Riemann zeta function. We have the functional equation

\[
    p_n(1-s) = \begin{cases} 
        p_n(s) & \text{if } n = 0, 1 \mod 4; \\
        p_n(s) & \text{if } n = 2, 3 \mod 4.
    \end{cases}
\]

Moreover

**Theorem 1.** The zeros of \( p_n \) lie on the line \( \text{Re}(s) = \frac{1}{2} \).

We give two proofs of this. Another proof may be found in Bump and Ng [5].

**First Proof.** We recall a familiar classical fact, that orthogonal polynomials have real zeros. More precisely, let \( \mu \) be a positive Borel measure on \( \mathbb{R} \), and assume that \( \mu \) is not supported on any finite set. We may apply Gram-Schmidt process to the sequence \( \{1, x, x^2, \ldots\} \) and obtain a sequence of polynomials \( P_0, P_1, P_2, \ldots \) such that the degree of \( P_n \) is \( n \), which are orthogonal with respect to \( \mu \). The zeros of these are real and simple. Indeed, after multiplying the polynomials \( P_n \) by suitable constants, they'll have real coefficients. If \( r_1, \ldots, r_k \) are the zeros of \( P_n \) which have odd multiplicity, if \( k < n \) we could expand \( Q(x) - \prod_{i=1}^k (x-r_i) \) in terms of \( P_i \) with \( i < n \), so \( Q \) would be orthogonal to \( P_n \), but patently \( Q P_n > 0 \), so this is a contradiction.

Let us show that the polynomials \( p_{2n} \left( \frac{1}{2} + it \right) \) form an orthogonal family with respect to a suitable measure. Indeed, the even Hermite functions \( f_{2n} \) are eigenfunctions of a self-adjoint differential operator (the oscillator Hamiltonian), so they form an orthogonal family on the half-line \( \mathbb{R}^+ \), which we parametrize exponentially. Thus, consider the functions \( \phi_n(x) = f_{2n}(e^{2\pi x}) e^{\pi x} \). These are orthogonal with respect to Lebesgue measure on \( \mathbb{R} \). The Fourier transform of \( \phi_n \) is \( 2\pi M_{2n} \left( \frac{1}{2} + it \right) \), so by the Plancherel theorem these are orthogonal:

\[
    \int_{-\infty}^{\infty} M_{2n} \left( \frac{1}{2} + it \right) M_{2m} \left( \frac{1}{2} + it \right) dt = 0
\]

if \( m \neq n \). Thus the polynomials \( p_{2n} \left( \frac{1}{2} + it \right) \) form an orthonormal family, with respect to the measure \( |1', \left( \frac{1}{2} + \frac{i}{2} \right)|^2 dt \).

Similarly, the polynomials \( p_{2n+1} \) are orthogonal with respect to \( |1', \left( \frac{3}{2} + \frac{i}{2} \right)|^2 dt \). They must therefore all have real zeros. 

**Second Proof.** Let \( f \) be an eigenfunction of the oscillator Hamiltonian. Thus, \( f \) satisfies the Schrödinger equation

\[
    \left( x^2 - \frac{1}{4\pi^2} \frac{d^2}{dx^2} \right) f = \frac{\lambda}{2\pi} f
\]

for some value of \( \lambda \). Define the Mellin transform

\[
    M(s) = \int_{0}^{\infty} f(x) x^s \frac{dx}{x}.
\]
Integrating the above Schrödinger equation by parts gives

\[ M(s + 2) - \frac{1}{4\pi^2} (s - 1)(s - 2) M(s - 2) = \frac{\lambda}{2\pi} M(s). \]

We have either

\[ M(s) = \begin{cases} 
\frac{n^{-s/2} \Gamma\left(\frac{s}{2}\right)}{\sqrt{2\pi}} p(s), \\
\frac{n^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right)}{\sqrt{2\pi}} \sqrt{2\pi} p(s),
\end{cases} \]

with \( p(s) \) a polynomial, according as \( \hat{f} = \perp f \) or \( \hat{f} = \perp if \) (i.e., according as \( f = f_n \) with \( n \) even or \( n \) odd.) We have therefore either

\[ \lambda p(s) = sp(s + 2) - (s - 1)p(s - 2), \]

or

\[ \lambda p(s) = (s + 1)p(s + 2) - (s - 2)p(s - 2). \]

The situation will be more symmetrical if we make the substitution \( q(s) = p(s + \frac{1}{2}) \).

Thus, we wish to show the zeros of \( q \) are purely imaginary, and we have

\[ \lambda q(s) = (s + a)q(s + 2) - (s - a)q(s - 2), \]

with \( a = \frac{3}{2} \). The theorem now follows from the following

**Lemma.** Let \( q(s) \) be a polynomial, and assume that the zeros of \( q(s) \) lie in the closed strip \( \{ \text{re}(s) \in [c, c] \} \) with \( c > 0 \). Then if \( a > 0 \), the zeros of

\[ r(s) = (s + a)q(s + 2) - (s - 2)q(s - 2) \]

lie in the open strip \( \{ \text{re}(s) \in (-c, c) \} \).

To prove this, suppose that \( \text{re}(s) \geq c \), yet \( r(s) = 0 \). We will obtain a contradiction. (The case \( \text{re}(s) \leq -c \) may be handled similarly.) Let \( q(s) = c \prod_{i=1}^{n} (s - r_i) \). If \( r(s) = 0 \), then

\[ |(s + a)q(s + 2)| - |(s - a)q(s - 2)|, \]

so

\[ |s + a| \prod |s + 2 - r_i| - |s - a| \prod |s - 2 - r_i|. \]

Now since \( \text{re}(s) > 0, a > 0 \), we have \( |s + a| > |s - a| \); moreover, since \( |\text{re}(r_i)| \leq c, \) \( \text{re}(s) > c \), we have \( \text{re}(s - r_i) > 0 \), and so \( |s + 2 - r_i| > |s - 2 - r_i| \). Multiplying these inequalities together, we obtain a contradiction. \( \blacksquare \)
The preceding proof is similar to the original proof of Pólya of an interesting property of the K-Bessel functions, namely, his theorem that if \( y > 0 \) and \( K_\nu(y) = 0 \), then \( \nu \) is purely imaginary. Pólya's proof [23] depends on the recurrence identity (Watson [34], 3.71)

\[
2\nu K_\nu(x) = x (K_{\nu+1}(x) - K_{\nu-1}(x)).
\]

The operator which takes an even function \( q(\nu) \) and replaces it by \( \nu^{-1} \left(q(\nu+1) - q(\nu-1)\right) \) has the property (like the operator \( q \to r \) in the Lemma) of moving the zeros of a function closer to the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis. Since \( \nu \to K_{\nu}(x) \) is not a polynomial function, making this argument rigorous requires care. An easier (but arguably less insightful) proof may be found in Titchmarsh [31], Section 10.23.

Pólya connects his result with the Riemann hypothesis by arguing that

\[
\pi^2 \left( K_{\nu+1}(2\pi) + K_{\nu-1}(2\pi) \right)
\]

has analytic properties similar to \( \frac{1}{s} \zeta(s) \) with \( s = \frac{1}{2} + it \). (Actually this value, taken from Titchmarsh [31], seems to us to be off by a constant, but this is unimportant.) This function also has its zeros on the line \( \text{re}(s) = \frac{1}{2} \).

It is worth pointing out that there is another more “philosophical” way of connecting Pólya’s result on the Bessel functions with the Riemann hypothesis. We begin by noting that it implies a Riemann hypothesis for the Fourier coefficients of Eisenstein series. Consider the classical \( SL(2, \mathbb{Z}) \) Eisenstein series

\[
E(z, s) = \frac{1}{2\pi} \Gamma(s) \sum_{|cz+d| \neq 0} \frac{y^s}{|d|^{1/2}},
\]

where the summation is over nonzero pairs of integers \((c, d)\). It is well known that if \( n \neq 0 \), then the \( n \)-th Fourier coefficient

\[
\int_0^1 E(x + iy) e^{2\pi i nx} \, dx = 2 |n|^{s-1/2} \sigma_{1-2s}(|n|) \sqrt{y} K_{s-1/2}(2\pi |n| y).
\]

(See Bump [2] Section 1.6.) Both the divisor function \( \sigma_{1-2s}(|n|) \) and the K-Bessel function \( K_{s-1/2} \) have their zeros on the line \( \text{re}(s) = \frac{1}{2} \). Now if, on the other hand, we consider the Eisenstein series of half-integral weight (see Maass [20], Shimura [28] and Goldfeld and Hoffstein [13]), the Fourier coefficients are quadratic L-functions. So the analogous assertion—that the Fourier coefficients of the Eisenstein series satisfy a Riemann hypothesis—in the case of the Eisenstein series of half-integral weight, should reduce to the classical Riemann hypothesis.

One may be a bit more careful here. Actually the Fourier coefficients of these Eisenstein series are the products of quadratic L-functions with certain finite Dirichlet polynomials, and one would like to assert that these polynomials themselves have their zeros
on the line \( \text{re}(s) = 1/2 \). David Cardon has looked at the case of Eisenstein series on
double cover of \( GL(2) \) over a rational function field, and his work suggests that the
formulation is that the Whittaker coefficients in the modified sense of Colmez, 

We propose here a conjectural generalization of Pólya's result on the zeros of the Bessel
function \( K_\nu \). Let \( \pi \) be a spherical principal series representation of \( PGL(2, \mathbb{R}) \), and let \( W \)
be the \( SO(2) \)-fixed vector (determined up to constant multiple) in its Whittaker model
with respect to the additive character \( \psi(x) = e^{2\pi i x} \) of \( \mathbb{R} \). Then

\[
W \left( \begin{pmatrix} y^{1/2} & xy & y^{-1/2} \\ \nu & 2\pi y & \nu \\ k \end{pmatrix} \right) \sim y^{\nu} K_\nu(2\pi y) e^{2i\pi y},
\]

when \( k \in SO(2) \), for some complex number \( \nu \). So Pólya's result may be formulated as
saying that if the \( SO(2) \)-fixed Whittaker vector in a spherical principal series representation
vanishes anywhere on \( PGL(2, \mathbb{R}) \), then the representation is tempered.

More generally, let \( \pi \) be a spherical principal series representation of \( PGL(n, \mathbb{R}) \),
and assume that \( \pi \) is a symmetric \( n-1 \)-st power lifting of a spherical principal series
representation of \( PGL(2, \mathbb{R}) \). This means that there is a quasicharacter \( \chi \) of \( \mathbb{R}^* / \{ \pm 1 \} \)
such that \( \pi \) is obtained by normalized parabolic induction from the character

\[
\begin{pmatrix}
\chi(y_1)^{n-1} & \chi(y_2)^{n-3} & \cdots & \chi(y_n)^{1-n}
\end{pmatrix}
\]

Let \( W \) be the \( SO(n) \)-fixed vector in the Whittaker model of \( \pi \), determined up to constant
multiple.

Conjecture. In this setting, if \( W \) vanishes anywhere on \( GL(n, \mathbb{R}) \), then \( \pi \) is tempered
(i.e. \( \chi \) is unitary).

We will offer three pieces of evidence for this statement.

Firstly, it is true when \( n = 2 \) by Pólya's result.

Secondly, for one particular nontempered spherical Whittaker function (which is a
symmetric square lift from \( GL(2) \)) on \( GL(3, \mathbb{R}) \) we can verify this claim—we recall that
the spherical Whittaker functions on \( GL(3, \mathbb{R}) \) and \( GL(3, \mathbb{C}) \) are the same, and that for
one particular principal series representation, corresponding to the cubic theta function
on \( GL(3, \mathbb{C}) \), the Whittaker function can be expressed in terms of the Bessel function
\( K_{1/3} \), so the asserted nonvanishing follows from Pólya's result. See Bump and Fried-
berg [3] and Bump and Huntley [4].

And thirdly, an analogous statement is true for spherical Whittaker functions on
\( PGL(n, F) \), when \( F \) is a nonarchimedean local field. Let \( \pi \) be a spherical principal series
representation with Satake parameters $\alpha_1, \cdots, \alpha_n$. Let
\[
h = \begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix}
\]
be a dominant element of the diagonal subgroup, so that if $\lambda_i$ is the valuation of $y_i$, we have $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Let $s_\lambda$ be the Schur polynomial corresponding to the partition $\lambda - (\lambda_1, \cdots, \lambda_n)$, a symmetric polynomial in $n$ variables (Macdonald [21]). According to Shimani [29] and Casselman and Shalika [8], the value $W(h)$ of the normalized Whittaker function with respect to an additive character $\psi$ whose conductor is the ring $\mathfrak{o}$ of integers in $F$ equals $\delta(h)^{1/2} s_\lambda(\alpha_1, \cdots, \alpha_n)$, where $\delta$ is the modular quasicharacter of the Borel subgroup of $GL(n, F)$. Now suppose that $\pi$ is a symmetric $n$-1-st power lift from $GL(2)$. Thus we assume that there exists a complex number $\alpha$ such that
\[
(\alpha_1, \cdots, \alpha_n) = (\alpha^{n-1}, \alpha^{n-2}, \cdots, \alpha^{1-n}).
\]

**Proposition.** In this situation, if $W(h) = 0$ for $h$ dominant, then $n$ is tempered.

**Proof.** We have $s_\lambda(\alpha^n, \alpha^{n^2}, \cdots, \alpha^{n^n}) = 0$, and we will show that $|\alpha| = 1$. Indeed, by homogeneity of the Schur polynomial, we have $s_\lambda(\alpha^{2n-2}, \alpha^{2n-4}, \cdots, 1) = 0$. We recall that
\[
s_\lambda(\alpha_1, \cdots, \alpha_n) = \begin{vmatrix}
\alpha_1^{\lambda_1+n-1} & \alpha_2^{\lambda_2+n-1} & \cdots & \alpha_n^{\lambda_n+n-1} \\
\alpha_1^{\lambda_2+n-2} & \alpha_2^{\lambda_2+n-2} & \cdots & \alpha_n^{\lambda_n+n-2} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{\lambda_n+n-2} & \alpha_2^{\lambda_n+n-2} & \cdots & \alpha_n^{\lambda_n+n-2} \\
\alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \\
\alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_n^{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{vmatrix}.
\]
Substituting $(\alpha^{2n-2}, \alpha^{2n-4}, \cdots, 1)$ for $(\alpha_1, \cdots, \alpha_n)$, the numerator here becomes
\[
\begin{vmatrix}
\beta_1^{n-1} & \beta_2^{n-1} & \cdots & 1 \\
\beta_2^{n-2} & \beta_2^{n-2} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
\beta_n^{n-1} & \beta_n^{n-1} & \cdots & 1
\end{vmatrix} = \prod_{i<j}(\beta_i - \beta_j),
\]
where $\beta_i = \alpha^{2(\lambda_i+n-i)}$. If this is zero, then some $\beta_i = \beta_j$, which implies that $\alpha$ is a root of unity. Thus $|\alpha| = 1$, so $n$ is tempered. \[\Box\]
2. The metaplectic representation. Witten, Brekke, Freund and Olsen in [1], [10] and [9] considered p-adic analogs of bosonic string theory. This led Ruelle, Thiran, Verstegen and Weyers [27] to consider the p-adic harmonic oscillator, also studied in the recent book of Vladimirov, Volovich and Zelenov [32]. The p-adic harmonic oscillator may be understood in terms of the restriction of the metaplectic representation of the double cover of $SL(2, \mathbb{R})$ on $L^2(\mathbb{R})$ to the group $SO(2)$ of symmetries of the Hamiltonian of a single particle moving in a quadratic potential field. In this formulation, there is no obstacle to replacing $\mathbb{R}$ by an arbitrary local field, and this is the point of view we will take.

Let $F$ be a local field of characteristic not equal to 2. Let $\langle \cdot, \cdot \rangle$ denote the Hilbert symbol of $F$. Let $\psi$ denote a nontrivial additive character of $F$. Let $dx$ denote the measure on $F$ which is self-dual with respect to the Fourier transform; thus if

$$\hat{f}(x) = \int_{F} f(y) \psi(2xy) \, dy,$$

$dx$ is self-dual if $\hat{f}(x) = f(-x)$. If $t \in F^\times$, let

$$\gamma(t) = |t|^{1/2} \int_{F} \psi(tx^2) \, dx.$$

This oscillatory integral is conditionally convergent in an obvious sense. The absolute value of $\gamma$ equals 1—and indeed it is an eight-th root of unity—and

$$\gamma(a) \gamma(b) = (a, b) \gamma(ab) \gamma(1).$$

Furthermore, we have

$$\gamma(b^2 a) = \gamma(a), \quad \gamma(a) = \gamma(a)^{-1}.$$

Let $G = SL(2, F)$, and let $\widetilde{G}$ be the metaplectic double cover of $SL(2, F)$ defined by Kubota's cocycle $\sigma : G \times G \to \mu_2 = \{\pm 1\}$. Thus in terms of the Hilbert symbol,

$$\sigma(g_1, g_2) = \begin{pmatrix} X(g_1) & X(g_2) \\ X(g_1 g_2) & X(g_1 g_2) \end{pmatrix},$$

where

$$X \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0; \\ d & \text{otherwise.} \end{cases}$$

Let $s : G \to \widetilde{G}$ be the standard section, so that

$$s(g_1) s(g_2) = \sigma(g_1, g_2) s(g_1 g_2).$$
We will also use the notation

$$[\begin{array}{cc} a & b \\
 c & d \end{array}] = s(\begin{array}{cc} a & b \\
 c & d \end{array}) \in \mathcal{G}.$$ 

The metaplectic representation $\omega_{\psi}$ is an action of $\mathcal{G}$ on the Schwartz space $S(F)$. It is given on generators by

$$\left( \omega \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right)(x) = \psi(tx^2) f(x),$$
$$\left( \omega \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \right)(x) = \gamma(1) f(x),$$
$$\left( \omega \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \right)(x) = |a|^{1/2} \frac{\gamma(1)}{\gamma(a)} f(ax).$$

See Weil [35] and Gelbart and Piatetski-Shapiro [12].

Let

$$H = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in F, a^2 + b^2 = 1 \right\},$$

and let $\tilde{H}$ be the preimage of $H$ in $\mathcal{G}$. Let $H'$ be the unique maximal compact subgroup of $H$, $\tilde{H}'$ its preimage in $\tilde{H}$. If $-1$ is not a square in $F$, then $H$ is compact, so actually $H' = H$ and $\tilde{H}' = \tilde{H}$. On the other hand, if $-1$ is a square, then $H \sim F^\times$, so $H'$ is a proper subgroup. The action of $\tilde{H}$ on the Schwartz space by means of the metaplectic representation is given by the following formula:

$$\left( \omega \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right)(x) = |b|^{-1/2} \frac{\gamma(b)}{\gamma(a)} \int_F \psi \left( \frac{1}{b}(ax^2 - 2xy + ay^2) \right) f(y) dy.$$ 

If $-1$ is not a square, so that $\tilde{H}$ is compact, then the restriction of $\omega$ to $\tilde{H}$ is multiplicity-free. If $F = \mathbb{R}$, this follows from our proof of Theorem 2 below (though it was known long before by Howe). If $F$ is $p$-adic, this follows from the Howe duality principle for the dual pair $U(1) \times U(1)$ in $SL(2)$. (Our group $SO(2)$ is the same as $U(1)$.) See Howe [16] and Waldspurger [33] for Howe duality, which is a theorem except in residual characteristic two. Other papers concerned specifically with the character of the metaplectic representation restricted to $SO(2)$ in the case of odd residual characteristic are Mocn [22] and Prasad [24]. Tonghai Yang [37] has formulas for the actual eigenfunctions of $U(1)$ acting on the Schwartz space.

In the case of residue characteristic two, the fact that the restriction of the metaplectic representation to compact $SO(2)$ is multiplicity-free is still known. This is implicit in the
work of Rogawski [26], which uses global to local methods, and a purely local proof may be found in Harris, Kudla and Sweet [14]. Also P. Ruelle, E. Thiran, D. Verstegen and J. Weyers [27] have calculated the character of the restriction of the metaplectic representation to tori in the fields \(\mathbb{Q}_p\), including \(\mathbb{Q}_2\), and their result implies this multiplicity one statement for \(\mathbb{Q}_2\).

On the other hand if \(-1\) is a square in \(F\), the restriction of \(\omega\) to \(\widetilde{H}\) does not decompose into a direct sum of constituents (though its dual space of distributions does so decompose). Instead we will consider the group \(\widetilde{H}'\). The restriction of \(\omega\) to this group is not multiplicity free.

The metaplectic cover splits over \(\widetilde{H}\). Indeed, if \(-1\) is not a square, \(\widetilde{H}\) is contained in \(SL(2, \mathbb{o})\), and an explicit splitting over this maximal compact subgroup was given by Kubota [18]. If we define

\[
\kappa \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) = \begin{cases} -1 & \text{if } v(b) \text{ is odd and } a \equiv -1 \text{ modulo } p; \\ 1 & \text{otherwise}, \end{cases}
\]

then

\[
\sigma(g_1, g_2) = \frac{\kappa(g_1) \kappa(g_2)}{\kappa(g_1, g_2)}
\]

when \(g_1, g_2 \in \mathbb{H}\). (It is worth mentioning that if the valuation \(v(b) > 0\), then \(a = \pm 1\) modulo \(p\) since \(a^2 + b^2 = 1\).) We may therefore define a representation of the abelian group \(\mathbb{H}\) by

\[
\left( \omega \left( \begin{array}{cc} a & b \\ b & a \end{array} \right) f \right)(x) = \kappa \left( \begin{array}{cc} a & b \\ b & a \end{array} \right) |b|^{1/2} \gamma(b)^{1/2} \int_{\mathbb{F}} \psi \left( \frac{1}{b} \left( ax^2 - 2xy + ay^2 \right) \right) f(y) dy.
\]

On the other hand, if \(-1\) is a square in \(F\), then \(H\) is conjugate to the diagonal torus in \(SL(2)\), and it is well known (and easy to prove from Kubota's cocycle formula) that the metaplectic cover splits over this subgroup. Since the cover splits over \(\mathbb{H}'\), we may regard \(\omega\) as giving a representation of this group.

**Local Riemann Hypothesis.** Suppose that \(F\) is a local field. Assume that \(F\) is not complex, and that the characteristic of \(F\) is not equal to 2. Let \(f \in S(F)\) be an eigenfunction of this action of \(H \cap K\), and let \(\nu\) be a character of \(F^\times\). Then the Mellin transform

\[
\int_{\mathbb{F}} f(x) \nu(x) |x|^s dx,
\]

if not identically zero, has its only zeros on the line \(\Re(s) = \frac{1}{2}\).

This assertion is largely proved, in this paper and its companion piece, Kurlberg [19].
Let us study what happens when we change the additive character. If \( \lambda \in F^\times \), let \( \psi_\lambda \) be the character \( x \mapsto \psi(\lambda x) \). Let \( d\psi_\lambda \) denote the additive Haar measure which is self-dual with respect to \( \psi \). Then \( d\psi_\lambda x = |\lambda|^{1/2} d\psi x \). Let \( \omega_\psi \) denote the metaplectic representation parametrized by \( \psi \). If \( f \in S(F) \), let \( f_\lambda(x) = f(\lambda x) \). Then it is easy to see that

\[
\omega_{\psi,2}\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f_\lambda\right)(x) = \omega_{\psi}\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f\right)(\lambda x).
\]

Thus if \( f \) is an eigenfunction of \( \tilde{\mathcal{H}} \) under the representation \( \omega_\psi \), then \( f_\lambda \) is an eigenfunction of \( \tilde{\mathcal{H}} \) under \( \omega_{\psi,2} \). The zeros of \( \zeta(s, \nu, f) \) and \( \zeta(s, \nu, f_\lambda) \) are at the same places, so we have the freedom to change \( \psi \) to \( \psi_\lambda \) for any square \( \lambda^2 \).

**Theorem 2.** The Local Riemann Hypothesis is true if \( F = \mathbb{R} \).

**Proof.** We reduce this to Theorem 1. Since we have the freedom to change \( \psi \) by a square, we may assume that \( \psi(x) = e^{+ix} \). We will assume that \( \psi(x) = e^{ix} \); the other case is obtained by replacing \( i \) by \(-i\) throughout the following discussion.

In this case, the self-dual measure on \( \mathbb{R} \) coincides with Lebesgue measure, and

\[
\gamma(1) = \int_{-\infty}^{\infty} e^{i \pi x^2} \, dx = \lim_{t \to 0^+} \int_{-\infty}^{\infty} e^{i \pi t x^2} \, dx = \lim_{t \to 0^+} (t - i)^{1/2} = \frac{1}{\sqrt{2}} (1 - i).
\]

Let \( g \) be the Lie algebra of \( SL(2, \mathbb{R}) \). The exponential map \( g \to SL(2, \mathbb{R}) \) lifts to a map \( \exp : g \to \hat{G} \). We then have a representation \( d\omega \) of \( g \) on \( S(\mathbb{R}) \) by

\[
((d\omega X)(f))(x) = \frac{d}{dt} (\exp(IX) f)(x)|_{t=0}.
\]

Let \( \hat{\mathcal{F}} : S(\mathbb{R}) \to S(\mathbb{R}) \) denote the Fourier transform \( \hat{\mathcal{F}}f = \hat{F} \), and let \( \hat{\mathcal{F}}^{-1} \) be its inverse:

\[
(\hat{\mathcal{F}}^{-1} f)(x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi ixy} \, dy.
\]

Define “momentum” and “position” operators \( P \) and \( Q \) on the Schwartz space by

\[
(Pf)(x) = \frac{1}{2\pi i} \frac{d}{dx} (x \, f(x)), \quad (Qf)(x) = x \, f(x).
\]

We have

\[
\hat{\mathcal{F}}^{-1} Q^2 \hat{\mathcal{F}} = \hat{t}^2.
\]
Indeed, \( \langle \mathfrak{F}^{-1} Q^2 \mathfrak{F} f \rangle(x) \) equals

\[
\int_{-\infty}^{\infty} y^2 \tilde{f}(y) e^{-2\pi ixy} dy - \frac{1}{4\pi^2} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} \tilde{f}(y) e^{-2\pi ixy} dy - \frac{1}{4\pi^2} \frac{d^2}{dx^2} f(x).
\]

We now prove that

\[
d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} i\pi Q^2, \quad d\omega \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} i\pi P^2.
\]

The first identity follows directly from the definitions:

\[
\left( d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f \right)(x) = \frac{d}{dt} \left( \omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} f \right)(x) \bigg|_{t=0} = \frac{d}{dt} e^{i\pi x^2} f(x) \bigg|_{t=0} = i\pi x^2 f(x).
\]

Since

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

we have

\[
d\omega \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \left( \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \left( d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \left( \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = i\pi \mathfrak{F}^{-1} Q^2 \mathfrak{F},
\]

and so the second identity follows from the first.

Now suppose that \( f \) is an eigenfunction of \( \tilde{H} \). Since

\[
\tilde{H} - e^{i\pi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\( f \) is also an eigenfunction of

\[
\frac{d}{dx} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{d}{dx} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{d}{dx} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = i\pi (P^2 + Q^2),
\]

which is (up to constant) the oscillator Hamiltonian. Hence \( f \) is one of the functions \( f_n \).

There are two possibilities for \( \nu(x) \cdot \text{sgn}(x)^\delta \), where \( \delta = 0 \) or \( 1 \). Depending on whether \( f \) is even or odd, exactly one of the integrals \( \int f(x) \nu(x) |x|^\delta dx/x \) will be nonzero, and this one will be just twice the Mellin transform of \( f \). Consequently, Theorem 2 follows from Theorem 1.
We turn now to the case of a $p$-adic field $F'$. In this case, following some preliminary investigation by Bump and Hoffstein, Kurlberg [19] has proved:

**Theorem 3.** The Local Riemann Hypothesis is true if $F'$ is a nonarchimedean local field of odd residual characteristic.

On the other hand, Kurlberg has also shown that the Local Riemann Hypothesis is false if $F = \mathbb{C}$.

3. Laguerre polynomials, the $n$-dimensional harmonic oscillator and a reciprocity law. The Laguerre polynomials (cf. Rainville [25]) are defined by:

\[ L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(x)^k}{k!} - \sum_{k=0}^{n} \frac{(1+\alpha)_n}{k!(n-k)!} \frac{(1+x)^k}{(1+\alpha)_k}, \]

where $(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1)$. They satisfy the differential equation

\[ x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (1+\alpha-x) \frac{d}{dx} L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0, \]

and the orthogonality relation:

\[ \int_{0}^{\infty} x^n e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{\Gamma(2+\alpha+n)} & \text{otherwise}. \end{cases} \]

Let $L_n^{(\alpha)}(x) = x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x)$. Then the Laguerre functions $L_n^{(\alpha)}$ are orthogonal with respect to Lebesgue measure on $[0, \infty)$. Their Mellin transforms

\[ M_n^{(\alpha)}(s) = \int_{0}^{\infty} L_n^{(\alpha)}(x) x^{s-1} \, dx = 2^{s+\frac{\alpha}{2}} \Gamma(s+\frac{\alpha}{2}) P_n^{(\alpha)}(s), \]

where

\[ P_n^{(\alpha)}(s) = \sum_{k=0}^{n} 2^k \binom{n+\alpha}{n-k} \binom{s+\alpha}{k/2}. \]

**Theorem 4.** The zeros of $P_n^{(\alpha)}(s)$ lie on the line $\text{re}(s) = \frac{1}{2}$.

**Proof.** The first proof of Theorem 1 is easily adapted. Using the orthogonality of the Laguerre functions, we see that the polynomials $P_n^{(\alpha)}(\frac{1}{2}+it)$ are orthogonal with respect to the measure $2^{1+\alpha} \Gamma^{2}[\frac{1}{2}+\frac{\alpha}{2}+it] \, dt$, and their zeros are therefore real. ■
The polynomials $P_n^{(x)}(s)$ satisfy a functional equation:

$$P_n^{(x)}(s) = (-1)^n P_n^{(x)}(1 - s).$$

We may prove this as follows. We start with the generating function for the Laguerre polynomials (Rainville [25], p. 202):

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 + t)^{-1-\alpha} e^{-x/(1-t)}.$$

Taking the Mellin transform in this identity yields

$$\sum_{n=0}^{\infty} P_n^{(\alpha)}(s) t^n = \alpha/2 (1 + t) - (1 - t)^{\alpha/2},$$

whence the functional equation.

Now let us investigate the harmonic oscillator in $n$-dimensions. If $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $r = |x| = \sqrt{\sum_i x_i^2}$ be the radial distance from the origin, and let $\Delta$ be the $n$-dimensional Laplacian $\sum_i \partial^2/\partial x_i^2$. Then consider the Schrödinger equation corresponding to a quadratic potential $V(r) = r^2$:

$$(\Delta + r^2) \psi = \epsilon \psi. \quad (4)$$

The eigenvalue $\epsilon$ is the energy level. The potential is rotationally symmetric and the Hamiltonian $-\Delta + r^2$ commutes with the orthogonal group. We may thus restrict ourselves to $\psi$ which lie in an irreducible subspace of $O(n)$.

**Theorem 5.** Let $\psi$ be a solution to (4) lying in an irreducible subspace of $O(n)$. Let $X$ be any radially symmetric function on $\mathbb{R}^n$, so that $X(tx) = X(x)$. Then the Mellin transform

$$\int_{\mathbb{R}^n} \phi(x) \frac{X(x)}{|x|^{2s}} \frac{1}{2\pi} \frac{dx}{dx}$$

has its zeros on the line $\text{Re}(s) = 1/2$.

**Proof.** We make use of spherical coordinates. Thus if $x \in \mathbb{R}^n$ is given, we take $r = |x| \in \mathbb{R}^+$ and $\xi = x/|x| \in S^{n-1}$ as basic coordinates. The group $O(n)$ acts on $L^2(S^{n-1})$, which decomposes as a direct sum of irreducible subspaces, each with multiplicity one. Because of this, our assumption that $\psi$ lies in an irreducible subspace of $O(n)$ implies that $\psi$ may be written in the form $\phi_0(r) \Phi(\xi)$, where $\Phi$ lies in one of these irreducible subspaces of $L^2(S^{n-1})$. Since $dx = r^{n-1} dr d\xi$, the integral equals

$$\int_0^\infty \phi_0(r) r^{2s+\frac{n}{2}-1} \frac{dr}{r}$$

for $s > n/4$.
times the inner product on $S^{n-1}$ of $X$ and $\Phi$. In spherical coordinates, the Laplacian in $n$ dimensions has the form:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Lambda,$$

where $\Lambda$ is the Laplacian on $S^{n-1}$ (Helgason, *Groups and Geometric Analysis* p.16). Moreover, the eigenvalue of $\Lambda$ on an element of an irreducible subspace of $S^{n-1}$ is equal to the eigenvalue of the Casimir operator on the corresponding irreducible representation, which Helgason shows has the form $-l(l + n - 2)$, where $l \in \mathbb{Z}$. We thus have the differential equation (with eigenvalue $\lambda$ for $\Lambda$):

$$\phi'' + \frac{n-1}{r} \phi' + \left( -\frac{l(l + n - 2)}{r^2} - r^2 + \epsilon \right) \phi = 0.$$

In order for $\phi_0 = e^{r^{2/3} r^l L(r^2)}$ to satisfy this differential equation, we need

$$r L'' + \left( l + \frac{n}{2} - r \right) L' + \left( \frac{c}{4} - \frac{l}{2} - \frac{n}{4} \right) L = 0.$$

This differential equation has a regular singular point at the origin, and a solution that is well-behaved there must be a constant multiple of $L^{-\frac{\lambda}{2} - \frac{3}{2}} L_k^{\frac{n+2-\lambda}{2}}$, where $k$ is an integer, and $\epsilon = 4k + 2l + n$. The result now follows from Theorem 4. ■

We note that this setup can be adapted to the metaplectic group by means of the Weil representation. The eigenfunctions at hand live in irreducible subspaces for the group $O(2) \times O(n)$, which is a maximal compact subgroup of the dual pair $SL(2, \mathbb{R}) \times O(n)$ in $Sp(2n, \mathbb{R})$, acting on $L^2(\mathbb{R}^n)$ via the standard polarization in the Weil representation. Expressed this way, the integrals of Theorem 5 have $p$-adic analogs, and though we haven't had a chance to investigate whether these satisfy a Riemann hypothesis, we hazard to conjecture that they do, at least in the case of anisotropic $O(n)$.

The polynomials $P_n^{(\alpha)}$ satisfy a reciprocity law relating their values at negative integers. We will show that

$$\begin{align*}
\binom{m + \alpha}{m} P_n^{(\alpha)}(-m - \frac{\alpha}{2}) = \binom{n + \alpha}{n} P_m^{(\alpha)}(-n - \frac{\alpha}{2}),
\end{align*}$$

Indeed, the left side equals

$$\sum_{k=0}^{n} 2^k \binom{m + \alpha}{m} \binom{n + \alpha}{n} \binom{m}{k},$$
and the reciprocity law follows from the identity
\[
\binom{m + \alpha}{m} \binom{n + \alpha}{n - k} \binom{m}{k} = \binom{n + \alpha}{n} \binom{m + \alpha}{m - k} \binom{n}{k}.
\]

We note the special case
(8) \[ P_n^{(0)}(-m) = P_m^{(0)}(-n). \]

This identity has an interesting combinatorial interpretation.

**Theorem 6.** \( P_n^{(0)}(-m) \) equals the number of lattice points \((x_1, \ldots, x_n) \in \mathbb{Z}^n\) such that \( \sum |x_i| \leq m \).

**Proof.** We can count the number of lattice points in \( \mathbb{Z}^n \) satisfying \( \sum |x_i| \leq m \) as follows. The number of lattice points having exactly \( k \) nonzero entries is \( 2^k \binom{n}{k} \binom{m}{k} \) if \( 0 \leq k \leq \min(m, n) \), because there are \( \binom{n}{k} \) choices for which coordinates shall be nonzero; and once this choice is fixed, there are \( 2^k \) possible distributions of signs, and \( \binom{m}{k} \) possible distributions of absolute values. Hence the number of lattice points is
\[
\sum_{k=0}^{\min(m, n)} 2^k \binom{n}{k} \binom{m}{k} P_n^{(0)}(-m).
\]

This completes the proof. \( \square \)

We derive a generating function for \( P_n^{(0)}(-m) \). Let \( a(m, n) \) be the number of lattice points satisfying the condition on the theorem. Then \( a(m, n) \) \( a(m, n + 1) \) is the number of lattice points satisfying exactly \( \sum |x_i| < m \) having a nonzero last component. If the last component is \( +m \) \( k \), with \( 0 \leq k \leq m - 1 \), then the number of possibilities for the first \( n - 1 \) components is \( a(k, n - 1) \), and so we have
\[
a(m, n) - a(m, n + 1) = 2 \sum_{k=0}^{m-1} a(k, n - 1).
\]

Hence (assuming \( m, n > 0 \)) we have
\[
a(m, n) \quad a(m, n + 1) \quad a(m, 1, n) \quad a(m, 1, n + 1) \quad 2a(m, 1, n + 1),
\]

which leads to the recursion
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a(m, n) x^m y^n = (1 - x - y - xy)^{-1}.
\]

The reciprocity law (8) is reflected by the symmetry of the generating function.
References


