Preliminary Notes on:

The Six Vertex Model and Alternating Sign Matrices

I Equivalence between Six-Vertex Configurations and Alternating Sign Matrices:

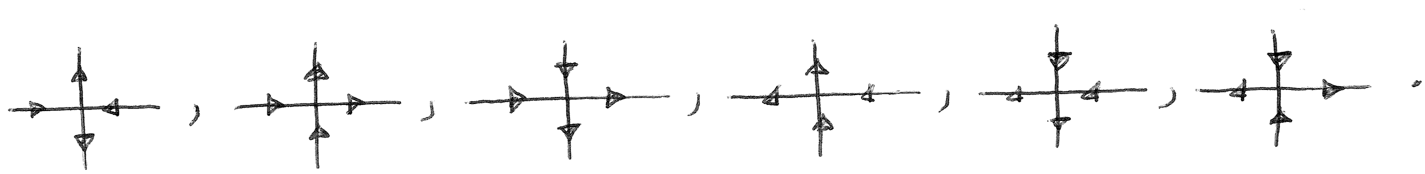
We start by giving the definition of these objects.

Definition: A six-vertex configuration of size N is an N by N square lattice where the directed edges satisfy the following constraints:

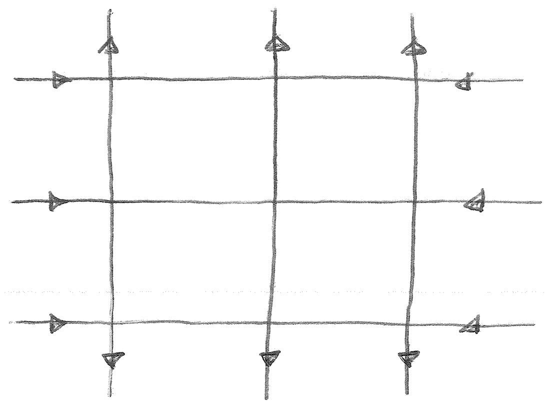
(i) each vertex has exactly two entering and two exiting edges;

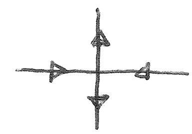
(ii) the domain wall boundary conditions are satisfied i.e. on the left and right boundaries all the edges are entering, on the top and bottom boundaries all the edges are exiting;

The condition (i) implies that only six configurations are valid at each vertex:

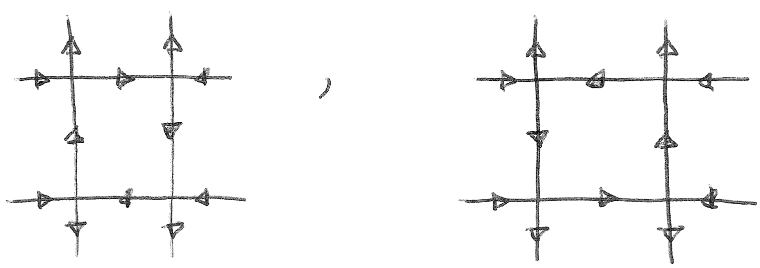


Here is an illustration of the domain wall boundary conditions on a configuration of size three.



There is only one configuration of size 1 :  .

There are two configurations of size 2 :



The set SV_N will denote the family of six-vertex configurations of size N. By inspection we find:

N	1	2	3	4	...
$\# SV_N$	1	2	7	42	...

Definition: An <u>alternating sign matrix</u> of size N is an N by N matrix $A = (a_{ij})$ with coefficients in $\{-1, 0, +1\}$ such that:

(i) for all $i, j \in [\![1, N]\!]$, $\sum_{i=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ij} = 1$;

(ii) non zero entries of A have alternating signs on lines and columns;

There is only one alternating sign matrix of size 1 : (1).

There are two alternating sign matrices of size 2:

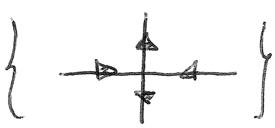
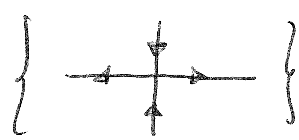
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The set ASM_N will denote the family of alternating sign matrices of size N. By inspection we find:

N	1	2	3	4	...
$\#ASM_N$	1	2	7	42	...

It turns out that alternating sign matrices and six vertex configurations of size N are equinumerous. Define the correspondence β between SV_N and ASM_N by:

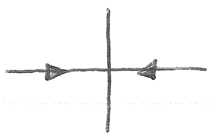
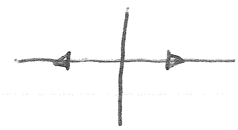
 $\xleftrightarrow{\beta}$ $\{0\}$;

 $\xleftrightarrow{\beta}$ $\{+1\}$ and  $\xleftrightarrow{\beta}$ $\{-1\}$.

<u>Theorem</u>: The correspondence $\beta : SV_N \longleftrightarrow ASM_N$ is a bijection.

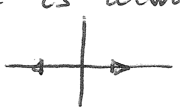
<u>Proof</u>: We describe what is going on for a line; mutatis mutandis the situation on the columns admits the same arguments.

We say that a vertex is an horizontal flip if the left and right edges have different orientations:

 and  .

On each lines the following properties are direct consequences of the domain wall boundary conditions:

① There is always an odd number of flips on each line;

② There is always one more flip of type  than of type  ;

③ The two types of flips alternate;

The correspondence β gives the translation of these properties in terms of a row in a matrix whose coefficients are in $\{-1, 0, +1\}$:

$\boxed{1}$ For all $i \in [\![1, N]\!]$, $\displaystyle\sum_{j=1}^{N} a_{ij} = \pm 1$;

$\boxed{2}$ There is always one more 1 tan -1 i.e. for all $i \in [\![1, N]\!]$ $\displaystyle\sum_{j=1}^{N} a_{ij} = +1$;

$\boxed{3}$ Non zero entries have alternating signs;

The same arguments apply on the columns hence $\beta : ASM_N \twoheadrightarrow SV_N$ is a surjective map.

The construction of a line of the lattice from a row of an alternating sign matrix goes as follows:

- start from the left boundary $\rightarrow$;
- as long as we encounter a 0 in the corresponding row keep the same orientation for the edge;
- every time we encounter a ± 1, we change the orientation.

This uniquely determine all the rows of the lattice configuration associated to an alternating sign matrix. The same construction applies on the columns.

Hence $\beta : ASM_N \hookrightarrow\!\!\!\!\!\twoheadrightarrow SV_N$ is injective, i.e. it is a bijection.

$\blacksquare$

Now that we have a bijection it is possible to use the properties of integravility of the six-vertex configurations to calculate the number of alternating sign matrices of seize N.

II Algebraic Bethe Ansatz for the Six Vertex Model:

The algebraic Bethe ansatz built from our R and L-operators can be linked to the partition function of lattice configurations.

Consider the space of states $St = V_1 \otimes \cdots \otimes V_N$ with $V_1 = \cdots = V_N = \mathbb{C}^2$ and the auxiliary space $A = \mathbb{C}^2$.

Define the R-operator of the system to be $R_{a_1,a_2}(\lambda) \in \mathrm{End}(A \otimes A)$ with λ a complex parameter such that on the canonical basis of $A \otimes A = \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$R_{a_1,a_2}(\lambda) \overset{\text{def}}{=} \begin{pmatrix} \lambda q - \lambda^{-1} q^{-1} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda q - \lambda^{-1} q^{-1} \end{pmatrix}.$$

Introduce the Pauli matrices that generates $sl_2(\mathbb{C})$:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to perform the calculations easily we rewrite the R-operator as:

$$\begin{aligned} R_{a_1,a_2}(\lambda) = {} & \frac{1}{2}(id_{a_1} + \sigma^z_{a_1}) \left[\frac{\lambda q - \lambda^{-1} q^{-1}}{2} (id_{a_2} + \sigma^z_{a_2}) + \frac{\lambda - \lambda^{-1}}{2} (id_{a_2} - \sigma^z_{a_2}) \right] \\ & + \frac{1}{2}(id_{a_1} - \sigma^z_{a_1}) \left[\frac{\lambda - \lambda^{-1}}{2} (id_{a_2} + \sigma^z_{a_2}) + \frac{\lambda q - \lambda^{-1} q^{-1}}{2} (id_{a_2} - \sigma^z_{a_2}) \right] \\ & + (q - q^{-1}) \left[\sigma^+_{a_1} \sigma^-_{a_2} + \sigma^-_{a_1} \sigma^+_{a_2} \right] \end{aligned}$$

with the usual tensor notations.

<u>Proposition:</u> The R-operator is a solution of the Yang-Baxter equation on $\mathrm{End}(A \otimes A \otimes A)$.

<u>Proof:</u> It is a calculation. We need to show that the following equation holds:

$$R_{a_1,a_2}(\lambda_1/\lambda_2) R_{a_1,a_3}(\lambda_1/\lambda_3) R_{a_2,a_3}(\lambda_2/\lambda_3)$$

$$= R_{a_2,a_3}(\lambda_2/\lambda_3) R_{a_1,a_3}(\lambda_1/\lambda_3) R_{a_1,a_2}(\lambda_1/\lambda_2) \quad .$$

This follows from our description of the R-operator based on the generators of $sl_2(\mathbb{C})$. We use the property that operators acting on different copies of the auxiliary space in the tensor product commute with each other:

$$T^{\bullet}_{a_i} T^{\bullet}_{a_j} = T^{\bullet}_{a_j} T^{\bullet}_{a_i} \qquad \text{if } i \neq j.$$

Then any of the 16 terms obtained by taking the product of the R-operators on the left hand side of Yang-Baxter equation is of the form:

$${}^{\bullet}_{a_1} {}^{\bullet}_{a_2} {}^{\bullet}_{a_1} {}^{\bullet}_{a_3} {}^{\bullet}_{a_2} {}^{\bullet}_{a_3}$$

since the components acting on the different spaces are factorized;

$$= {}^{\bullet}_{a_2} {}^{\bullet}_{a_1} {}^{\bullet}_{a_1} {}^{\bullet}_{a_3} {}^{\bullet}_{a_2} {}^{\bullet}_{a_3}$$

$$= \cdots = {}^{\bullet}_{a_2} {}^{\bullet}_{a_3} {}^{\bullet}_{a_1} {}^{\bullet}_{a_3} {}^{\bullet}_{a_1} {}^{\bullet}_{a_2}$$

simply rearranging with the use of the commutation property.

As a consequence any term of the left hand side of Yang-Baxter equation can be rearranged in a term of the right hand side proving the relation.

■

The L-operators $L_{a,n}(\lambda) \in \mathrm{End}(A \otimes \mathcal{S}t)$ for all $n \in [\![1, N]\!]$ are directly defined from the R-operator:

$$L_{a,n}(\lambda) \overset{def}{=} R_{a,n}(\lambda).$$

__Proposition:__ The L-operators satisfy the intertwining and ultralocality equation on $\mathrm{End}(A \otimes A \otimes \mathcal{S}t)$.

__Proof:__ The intertwining equation reads exactly like the Yang-Baxter equation.

The ultralocality equation is again just a consequence of the factorization of the action of the R-operator on the different factors of the tensor product, implying that everything commutes.

■

As usual the monodromy of the system $T_a(\lambda) \in \mathrm{End}(A \otimes \mathcal{S}t)$ is:

$$T_a(\lambda) \overset{def}{=} L_{a,N}(\lambda) \cdots L_{a,1}(\lambda).$$

By using the decomposition $\mathrm{End}(A \otimes \mathcal{S}t) \cong M_2(\mathbb{C}) \otimes \mathrm{End}(\mathcal{S}t)$ we write the monodromy as a matrix whose coefficients are operators acting on $\mathcal{S}t$:

$$T_a(\lambda) \overset{def}{=} \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{N}_+(\lambda) \\ \mathcal{N}_-(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} \quad \text{and} \quad \mathcal{Q}(\lambda) \overset{def}{=} tr_a T_a(\lambda) = \mathcal{A}(\lambda) + \mathcal{D}(\lambda).$$

Because of the relations satisfied by the L-operators, the monodromy automatically satisfies an intertwining equation:

$$R_{a_1,a_2}(\lambda/\mu)\, T_{a_1}(\lambda)\, T_{a_2}(\mu) = T_{a_2}(\mu)\, T_{a_1}(\lambda)\, R_{a_1,a_2}(\lambda/\mu).$$

Lemma: The operators $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{N}_+$ acting on St satisfy the following relations:

$$\mathcal{N}_+(\lambda)\mathcal{N}_+(\mu) = \mathcal{N}_+(\mu)\mathcal{N}_+(\lambda),$$

$$\mathcal{A}(\lambda)\mathcal{N}_+(\mu) = \frac{f(\mu/\lambda)}{g(\mu/\lambda)}\mathcal{N}_+(\mu)\mathcal{A}(\lambda) - \frac{h(\mu/\lambda)}{g(\mu/\lambda)}\mathcal{N}_+(\lambda)\mathcal{A}(\mu),$$

$$\mathcal{D}(\lambda)\mathcal{N}_+(\mu) = \frac{f(\lambda/\mu)}{g(\lambda/\mu)}\mathcal{N}_+(\mu)\mathcal{D}(\lambda) - \frac{h(\lambda/\mu)}{g(\lambda/\mu)}\mathcal{N}_+(\lambda)\mathcal{D}(\mu),$$

with $f(\lambda) = \lambda q - \lambda^{-1}q^{-1}$, $g(\lambda) = \lambda - \lambda^{-1}$, $h(\lambda) = q - q^{-1}$.

Proof: We can read these relations on the explicit form of the intertwining equation satisfied by the monodromy.

Consider $R_{a_1,a_2}(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & g(\lambda) & h(\lambda) & 0 \\ 0 & h(\lambda) & g(\lambda) & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix}$. Then the intertwining equation for the monodromy is:

$$\begin{pmatrix} * & * & f(\lambda/\mu)\, N_+(\lambda)\, A(\mu) & f(\lambda/\mu)\, N_+(\lambda) N_+(\mu) \\ * & * & * & * \\ * & * & * & h(\lambda/\mu)\, N_+(\lambda) D(\mu) + g(\lambda/\mu)\, D(\lambda) N_+(\mu) \\ * & * & * & * \end{pmatrix}$$

$$= \begin{pmatrix} * & * & h(\lambda/\mu)\, N_+(\mu)\, A(\lambda) + g(\lambda/\mu)\, A(\mu) N_+(\lambda) & f(\lambda/\mu)\, N_+(\mu) N_+(\lambda) \\ * & * & * & * \\ * & * & * & f(\lambda/\mu)\, N_+(\mu) D(\lambda) \\ * & * & * & * \end{pmatrix}.$$

The identification of the coefficients gives the desired equations. ∎

To apply the algebraic Bethe ansatz we need a vacuum vector. To find it lets look at $L_{a,m}(\lambda)$ written as an element of $M_2(\mathbb{C}) \otimes \mathrm{End}(St)$:

$$L_{a,m}(\lambda) = \begin{pmatrix} \hat{a}_m(\lambda) & \hat{\eta}_{+,m}(\lambda) \\ \hat{\eta}_{-,m}(\lambda) & \hat{d}_m(\lambda) \end{pmatrix} \quad \text{with} \quad \hat{a}_m(\lambda) = \begin{pmatrix} f(\lambda) & 0 \\ 0 & g(\lambda) \end{pmatrix}$$

$$\hat{d}_m(\lambda) = \begin{pmatrix} g(\lambda) & 0 \\ 0 & f(\lambda) \end{pmatrix}$$

$$\hat{\eta}_{+,m}(\lambda) = \begin{pmatrix} 0 & 0 \\ h(\lambda) & 0 \end{pmatrix} \quad \text{and} \quad \hat{\eta}_{-,m}(\lambda) = \begin{pmatrix} 0 & h(\lambda) \\ 0 & 0 \end{pmatrix}.$$

In other words, these operators act on the canonical basis of $\mathbb{C}^2$ as:

$$\begin{cases} \hat{a}_\circ(\lambda)|\uparrow\rangle = f(\lambda)|\uparrow\rangle \\ \hat{a}_\circ(\lambda)|\downarrow\rangle = g(\lambda)|\downarrow\rangle \end{cases}, \qquad \begin{cases} \hat{\eta}_{+,\circ}(\lambda)|\uparrow\rangle = h(\lambda)|\downarrow\rangle \\ \hat{\eta}_{+,\circ}(\lambda)|\downarrow\rangle = 0 \end{cases},$$

$$\begin{cases} \hat{d}_\circ(\lambda)|\uparrow\rangle = g(\lambda)|\uparrow\rangle \\ \hat{d}_\circ(\lambda)|\downarrow\rangle = f(\lambda)|\downarrow\rangle \end{cases}, \qquad \begin{cases} \hat{\eta}_{-,\circ}(\lambda)|\uparrow\rangle = 0 \\ \hat{\eta}_{-,\circ}(\lambda)|\downarrow\rangle = h(\lambda)|\uparrow\rangle \end{cases}.$$

$$\mathcal{A}(\lambda)\prod_{k=1}^{M}\mathcal{N}_+(\lambda_k)|\omega\rangle = \alpha(\lambda)\prod_{j=1}^{M}\frac{f(\lambda_j/\lambda)}{g(\lambda_j/\lambda)}\prod_{k=1}^{M}\mathcal{N}_+(\lambda_k)|\omega\rangle$$
$$+\sum_{i=1}^{M} -\alpha(\lambda_i)\frac{h(\lambda_i/\lambda)}{g(\lambda_i/\lambda)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{f(\lambda_j/\lambda_i)}{g(\lambda_j/\lambda_i)}\prod_{\substack{k=1\\ k\neq i}}^{M}\mathcal{N}_+(\lambda_k)|\omega\rangle$$

and

$$\mathcal{D}(\lambda)\prod_{k=1}^{M}\mathcal{N}_+(\lambda_k)|\omega\rangle = \delta(\lambda)\prod_{j=1}^{M}\frac{f(\lambda/\lambda_j)}{g(\lambda/\lambda_j)}\prod_{k=1}^{M}\mathcal{N}_+(\lambda_k)|\omega\rangle$$
$$+\sum_{i=1}^{M} -\delta(\lambda_i)\frac{h(\lambda/\lambda_i)}{g(\lambda/\lambda_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{f(\lambda_i/\lambda_j)}{g(\lambda_j/\lambda_i)}\prod_{\substack{k=1\\ k\neq i}}^{M}\mathcal{N}_+(\lambda_k)|\omega\rangle .$$

Hence, if we want $\mathcal{N}_+(\lambda_1)\cdots\mathcal{N}_+(\lambda_M)|\omega\rangle$ to be an eigenvector of $\mathcal{Q}(\lambda)$ with eigenvalue:

$$\alpha(\lambda)\prod_{j=1}^{M}\frac{f(\lambda_j/\lambda)}{g(\lambda_j/\lambda)} + \delta(\lambda)\prod_{j=1}^{M}\frac{f(\lambda/\lambda_j)}{g(\lambda/\lambda_j)},$$

we need that for all $i\in[\![1,M]\!]$

$$\alpha(\lambda_i)\frac{h(\lambda_i/\lambda)}{g(\lambda_i/\lambda)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{f(\lambda_j/\lambda_i)}{g(\lambda_j/\lambda_i)} + \delta(\lambda_i)\frac{h(\lambda/\lambda_i)}{g(\lambda/\lambda_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{f(\lambda_i/\lambda_j)}{g(\lambda_i/\lambda_j)} = 0 .$$

But since h is a constant and $g(\lambda/\mu) = -g(\mu/\lambda)$ it simplifies to Bethe equation:

$$\alpha(\lambda_i)\prod_{\substack{j=1\\ j\neq i}}^{M} f(\lambda_j/\lambda_i) = \delta(\lambda_i)\prod_{\substack{j=1\\ j\neq i}}^{M} f(\lambda_i/\lambda_j) . \quad \blacksquare$$

Define $|\omega\rangle \in \mathcal{H}$ to be the state of the system with all the spins up : $|\omega\rangle = |\uparrow \cdots \uparrow\rangle$.

<u>Proposition:</u> The vector $|\omega\rangle$ is a vacuum state.

<u>Proof:</u> It follows directly from the definition of the L-operators that:

$$\mathcal{A}(\lambda)|\omega\rangle = \alpha(\lambda)|\omega\rangle = f^N(\lambda)|\omega\rangle, \ \mathcal{D}(\lambda)|\omega\rangle = \delta(\lambda)|\omega\rangle = g^N(\lambda)|\omega\rangle, \ \mathcal{N}_-(\lambda)|\omega\rangle = 0 .$$

This is the definition of a vacuum state. ∎

<u>Theorem:</u> (Algebraic Bethe Ansatz) The vector $\mathcal{N}_+(\lambda_1)\cdots\mathcal{N}_+(\lambda_M)|\omega\rangle$ is an eigenvector of $\mathcal{Q}(\lambda)$ with eigenvalue:

$$\alpha(\lambda) \prod_{j=1}^{M} \frac{f(\lambda_j/\lambda)}{g(\lambda_j/\lambda)} + \delta(\lambda) \prod_{j=1}^{M} \frac{f(\lambda/\lambda_j)}{g(\lambda/\lambda_j)}$$

if and only if for all $i \in [\![1, M]\!]$, λ_i is solution of Bethe equation:

$$\frac{\alpha(\lambda_i)}{\delta(\lambda_i)} \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{f(\lambda_j/\lambda_i)}{f(\lambda_i/\lambda_j)} = 1 .$$

<u>Proof:</u> The algebraic Bethe ansatz consists in finding the conditions for $\mathcal{N}_+(\lambda_1)\cdots\mathcal{N}_+(\lambda_M)|\omega\rangle$ to be an eigenvector of $\mathcal{Q}(\lambda)$.

Since $\mathcal{Q}(\lambda) = \mathcal{A}(\lambda) + \mathcal{D}(\lambda)$ we get from the previous lemma:

In order to understand the link between the algebraic Bethe ansatz and the enumeration of lattice configurations it is necessary to look at the form of the eigenvectors $\mathcal{N}_+(\lambda_1)\cdots\mathcal{N}_+(\lambda_M)|\omega\rangle$.

First thing rewrite the parameters of the L-operators such that they keep track of the local space of states on which they are acting:

$$L_{a,m}(\lambda) \longmapsto L_{a,m}(\lambda/\mu_m).$$

With the notation $\underline{\mu}=\{\mu_1,\dots,\mu_N\}$ the monodromy of the system becomes:

$$T_a(\lambda,\underline{\mu}) = L_{a,N}(\lambda/\mu_N)\cdots L_{a,1}(\lambda/\mu_1) = \begin{pmatrix} \mathcal{A}(\lambda,\underline{\mu}) & \mathcal{N}_+(\lambda,\underline{\mu}) \\ \mathcal{N}_-(\lambda,\underline{\mu}) & \mathcal{D}(\lambda,\underline{\mu}) \end{pmatrix}.$$

Let $[L_{a,m}(\lambda/\mu_m)]_{pq}$ denote the (p,q) coefficient of the matrix $L_{a,m}(\lambda/\mu_m)\in M_2(\mathbb{C})\otimes \mathrm{End}(\mathcal{S}t)$. Then taking explicitly the product of the L-operators we obtain:

$$\mathcal{N}_+(\lambda,\underline{\mu}) = \sum_{i_1,\dots,i_{N-1}\in\{1,2\}} [L_{a,N}(\lambda/\mu_N)]_{1i_1}[L_{a,N-1}(\lambda/\mu_{N-1})]_{i_1i_2}\cdots[L_{a,1}(\lambda/\mu_1)]_{i_{N-1}2}.$$

Take $M=1$ and consider the action of $\mathcal{N}_+(\lambda_1,\underline{\mu})$ on $|\omega\rangle$:

$$\mathcal{N}_+(\lambda_1,\underline{\mu})|\uparrow\cdots\uparrow\rangle = \sum_{i_1,\dots,i_{N-1}\in\{1,2\}} [L_{a,N}(\lambda_1/\mu_N)]_{1i_1}\cdots[L_{a,1}(\lambda_1/\mu_1)]_{i_{N-1}2}|\uparrow\cdots\uparrow\rangle.$$

Since $[L_{a,\circ}(\circ)]_{21}|\uparrow\rangle = \hat{\eta}_{-,\circ}(\circ)|\uparrow\rangle = 0$ the only terms of the sum that are not equal to zero are the ones avoiding any matrix coefficient $(i_m, i_{m+1}) = (2,1)$.

Moreover the sequence of matrix coefficients in any term of the sum has the form:

$$(1 i_1)\,(i_1 i_2) \cdots\cdots (i_{N-2} i_{N-1})(i_{N-1} 2)$$

so that going from 1 to 2 there is at least one pair $(i_m, i_{m+1}) = (1,2)$, which corresponds to the following action:

$$[L_{a,0}(0)]_{12} |\uparrow\rangle = \hat{\eta}_{+,0}(0)|\uparrow\rangle = h(0)|\downarrow\rangle \; .$$

But there cannot be more than one pair $(i_m, i_{m+1}) = (1,2)$ as if it was the case we would find the following pattern in the sequence of matrix coefficients:

$$(1 i_1) \cdots\cdots (12) \cdots\cdots (12) \cdots\cdots (i_{N-1} 2)$$

but between two (12) there must be a (21) coefficient which anhilates the term.

Hence the action of $\mathcal{N}_+$ on $|\omega\rangle$ generates all the states of the system with exactly one spin down:

$$\mathcal{N}_+(\lambda_1, \underline{\mu}) = g(\lambda_1/\mu_N) \cdots g(\lambda_1/\mu_2)\, h(\lambda_1/\mu_1)\, |\downarrow\uparrow \cdots \uparrow\rangle$$

$$+ \; g(\lambda_1/\mu_N) \cdots g(\lambda_1/\mu_3)\, h(\lambda_1/\mu_2)\, f(\lambda_1/\mu_1)\, |\uparrow\downarrow\uparrow \cdots \uparrow\rangle$$

$$+ \cdots$$

$$+ \; h(\lambda_1/\mu_N)\, f(\lambda_1/\mu_{N-1}) \cdots f(\lambda_1/\mu_1)\, |\uparrow \cdots \uparrow\downarrow\rangle \; .$$

Explicitly for $N=3$:

$$\mathcal{N}_+(\lambda_1,\underline{\mu})|\omega\rangle = (q-q^{-1}) f(\lambda_1/\mu_2) f(\lambda_1/\mu_3) |\downarrow\uparrow\uparrow\rangle$$
$$+ \; g(\lambda_1/\mu_1) (q-q^{-1}) f(\lambda_1/\mu_3) |\uparrow\downarrow\uparrow\rangle$$
$$+ \; g(\lambda_1/\mu_1) g(\lambda_2/\mu_2) (q-q^{-1}) |\uparrow\uparrow\downarrow\rangle \quad .$$

Take $M=2$, then we can calculate $\mathcal{N}_+(\lambda_2,\underline{\mu})\mathcal{N}_+(\lambda_1,\underline{\mu})|\omega\rangle$ from $\mathcal{N}_+(\lambda_1,\underline{\mu})|\omega\rangle$. All we need to do is understand the action of $\mathcal{N}_+(\lambda_2,\underline{\mu})$ on the state $|\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow\rangle$ where there is exactly one spin down on the site p. So we consider:

$$\mathcal{N}_+(\lambda_2,\underline{\mu})|\uparrow\cdots\uparrow\overset{\overset{p}{\vee}}{\downarrow}\uparrow\cdots\uparrow\rangle = \sum_{i_1,\dots,i_{N-1}\in\{1,2\}} [L_{a,N}(\lambda_2/\mu_N)]_{1 i_1} \cdots [L_{a,1}(\lambda_2/\mu_1)]_{i_{N-1} 2} |\uparrow\cdots\uparrow\overset{\overset{p}{\vee}}{\downarrow}\uparrow\cdots\uparrow\rangle .$$

Since we have two kind of spins (up and down) we need to take care of two kind of actions that anhilates the vector $|\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow\rangle$:

$$[L_{a,\circ}(\circ)]_{21}|\uparrow\rangle = \hat{\eta}_{-,\circ}(\circ)|\uparrow\rangle = 0 \qquad \text{for all sites except } p ,$$

$$[L_{a,\circ}(\circ)]_{12}|\downarrow\rangle = \hat{\eta}_{+,\circ}(\circ)|\downarrow\rangle = 0 \qquad \text{only for site } p .$$

So there are only two possibilities:

(*) if (i_p, i_{p+1}) is not (21) or (12) there is exactly one $(i_n, i_{n+1}) = (12)$ in the sequence of coefficients;

(**) if $(i_p, i_{p+1}) = (21)$ then the sequence of matrix coefficients is $(1 i_1)\cdots(21)\cdots(i_{N-1}2)$ and in order to have a nonzero term we must find exactly one (12) between $(1i_1)$ and (21) and exactly one (12) between (21) and $(i_{N-1}2)$;

Since $[L_{a,0}(0)]_{21}\,|\downarrow\rangle = \hat{\eta}_{-,0}(0)|\downarrow\rangle = h(0)|\uparrow\rangle$ the result is the same in both cases, we end up with all the possible states having exactly two spins down.

Hence the action of $\mathcal{N}_+$ twice on $|\omega\rangle$ generates all the states of the system having exactly two spins down.

Here is what we get for a chain of length 2 (i.e. $N=2$):

- the monodromy gives $\mathcal{N}_+(\lambda_i,\underline{\mu}) = \hat{a}_2(\lambda_i/\mu_2)\,\hat{\eta}_{+,1}(\lambda_i/\mu_1) + \hat{\eta}_{+,2}(\lambda_i/\mu_2)\,\hat{d}_1(\lambda_i/\mu_1)$;

- the action of $\mathcal{N}_+$ on $|\omega\rangle = |\uparrow\uparrow\rangle$ gives

$$\mathcal{N}_+(\lambda_1,\underline{\mu})\,|\uparrow\uparrow\rangle = (q-q^{-1})\,f(\lambda_1/\mu_2)\,|\downarrow\uparrow\rangle + g(\lambda_1/\mu_1)\,(q-q^{-1})\,|\uparrow\downarrow\rangle\ ;$$

- the action of $\mathcal{N}_+$ twice on $|\omega\rangle$ is calculated from

$$\mathcal{N}_+(\lambda_2,\underline{\mu})\,|\downarrow\uparrow\rangle = g(\lambda_2/\mu_1)\,(q-q^{-1})\,|\downarrow\downarrow\rangle$$

$$\mathcal{N}_+(\lambda_2,\underline{\mu})\,|\uparrow\downarrow\rangle = (q-q^{-1})\,f(\lambda_2/\mu_2)\,|\downarrow\downarrow\rangle$$

hence

$$\mathcal{N}_+(\lambda_2,\underline{\mu})\,\mathcal{N}_+(\lambda_1,\underline{\mu})|\omega\rangle = \left[(q-q^{-1})^2 f(\lambda_1/\mu_2)\,g(\lambda_2/\mu_1) + (q-q^{-1})^2 g(\lambda_1/\mu_1)\,f(\lambda_2/\mu_2)\right]|\downarrow\downarrow\rangle .$$

With the same argument we conclude that $\mathcal{N}_{+}(\lambda_1, \underline{\mu}) \cdots \mathcal{N}_{+}(\lambda_M, \underline{\mu}) |\omega\rangle$ is a linear combination of all the states of the system with exactly M spins down for $M \in [\![1, N]\!]$ and 0 if $M > N$.

The next step is to interpret the coefficients in those linear combinations in terms of a generating function of lattice configurations.

III Partition Function of the Six Vertex Model

The key to derive an explicit formula for the partition function of the six vertex configurations is to reinterpret the algebraic Bethe ansatz.

We extend the definition of the L-operators to the tensor product

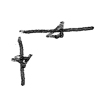
$$A_v \otimes St = \underbrace{A_1 \otimes \cdots \otimes A_N}_{= A_v} \otimes \underbrace{V_1 \otimes \cdots \otimes V_N}_{= St}$$

with $A_1 = \cdots = A_N = V_1 = \cdots = V_N = \mathbb{C}^2$ simply by considering $L_{a_i,m}(\lambda_i/\mu_m) = R_{a_i,m}(\lambda_i/\mu_m) \in \mathrm{End}(A_v \otimes St)$ as acting non trivially only on $A_i \otimes V_m$.

We identify $A_i \otimes V_m = \mathbb{C}^2 \otimes \mathbb{C}^2$ with the vector space corresponding to the local space of states of the configurations of the six vertex model at the intersection of line i and column m of the lattice.

For each $A_\circ \otimes V_\circ$ there is the following correspondence between the elements of the canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the oriented edges of the lattice:

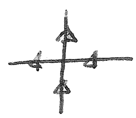
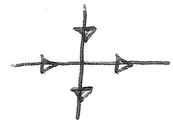
$|\uparrow\uparrow\rangle \longleftrightarrow$  or  , $|\downarrow\downarrow\rangle \longleftrightarrow$ or  ,

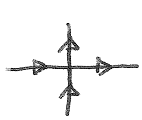
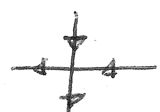
$|\uparrow\downarrow\rangle \longleftrightarrow$  or  , $|\downarrow\uparrow\rangle \longleftrightarrow$  or  .

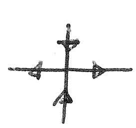
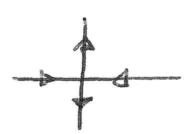
The L-operator acting on $A_i \otimes V_m$ is:

$$L_{a_i,m}(\lambda_i/\mu_m) = \begin{pmatrix} f(\lambda_i/\mu_m) & 0 & 0 & 0 \\ 0 & g(\lambda_i/\mu_m) & h(\lambda_i/\mu_m) & 0 \\ 0 & h(\lambda_i/\mu_m) & g(\lambda_i/\mu_m) & 0 \\ 0 & 0 & 0 & f(\lambda_i/\mu_m) \end{pmatrix}.$$

And we interpret it as giving the following weights to the vertex configurations at the intersection of line i and column m:

$f(\lambda_i/\mu_m) \longleftrightarrow$  and  ,

$g(\lambda_i/\mu_m) \longleftrightarrow$  and  ,

$h(\lambda_i/\mu_m) \longleftrightarrow$  and  ,

and 0 to the configurations that are not admissible for the six vertex model.

There is now a monodromy for each line of the lattice:

$$T_{a_i}(\lambda_i,\underline{\mu}) = L_{a_i,N}(\lambda_i/\mu_N)\cdots L_{a_i,1}(\lambda_i/\mu_1) = \begin{pmatrix} \mathcal{A}(\lambda_i,\underline{\mu}) & \mathcal{N}_+(\lambda_i,\underline{\mu}) \\ \mathcal{N}_-(\lambda_i,\underline{\mu}) & \mathcal{D}(\lambda_i,\underline{\mu}) \end{pmatrix}$$

for all $i \in [\![1,N]\!]$.

Property: The monodromies of the lines of the lattice satisfy intertwining equations on $\mathbb{E}\mathrm{nd}(A_0 \otimes \mathcal{S}t)$:

$$R_{a_i,a_{i+1}}(\lambda_i/\lambda_{i+1})\, T_{a_i}(\lambda_i,\underline{\mu})\, T_{a_{i+1}}(\lambda_{i+1},\underline{\mu})$$

$$= T_{a_{i+1}}(\lambda_{i+1},\underline{\mu})\, T_{a_i}(\lambda_i,\underline{\mu})\, R_{a_i,a_{i+1}}(\lambda_i/\lambda_{i+1})\,.$$

Proof: We have proved that already in part II. It still holds if we have more copies of the auxiliary space. $\blacksquare$

With this lattice interpretation we can also associate a monodromy to each column:

$$\widetilde{T}_m(\underline{\lambda},\mu_m) = L_{a_N,m}(\lambda_N/\mu_m)\cdots L_{a_1,m}(\lambda_1/\mu_m) = \begin{pmatrix} \widetilde{\mathcal{A}}(\underline{\lambda},\mu_m) & \widetilde{\mathcal{N}}_+(\underline{\lambda},\mu_m) \\ \widetilde{\mathcal{N}}_-(\underline{\lambda},\mu_m) & \widetilde{\mathcal{D}}(\underline{\lambda},\mu_m) \end{pmatrix}$$

for all $m \in [\![1,N]\!]$.

Since $A_0 = \mathcal{S}t$ and $R_{a_1,a_2}(\lambda)$, $L_{a,m}(\lambda)$ are symmetric with respect to their action on A_1, A_2 and A, V_m the monodromies of the columns also satisfy intertwining equations on $\mathbb{E}\mathrm{nd}(A_0 \otimes \mathcal{S}t)$:

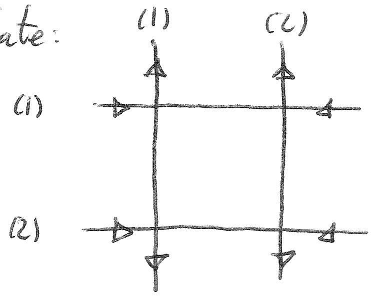
$$R_{m,m+1}(\mu_m/\mu_{m+1})\, \widetilde{T}_m(\underline{\lambda},\mu_m)\, \widetilde{T}_{m+1}(\underline{\lambda},\mu_{m+1})$$

$$= \widetilde{T}_{m+1}(\underline{\lambda},\mu_{m+1})\, \widetilde{T}_m(\underline{\lambda},\mu_m)\, R_{m,m+1}(\mu_m/\mu_{m+1})\,.$$

This implies that $\tilde{\mathcal{A}}, \tilde{\mathcal{N}}_+, \tilde{\mathcal{N}}_-, \tilde{\mathcal{D}}$ satisfy the same relations as $\mathcal{A}, \mathcal{N}_+, \mathcal{N}_-, \mathcal{D}$.

In order to understand how to get a generating function from the algebraic Bethe ansatz we give the construction of the six vertex configurations of size $N=2$ line by line.

Starting with the domain wall boundary conditions as the initial state:

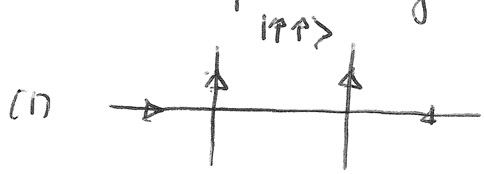


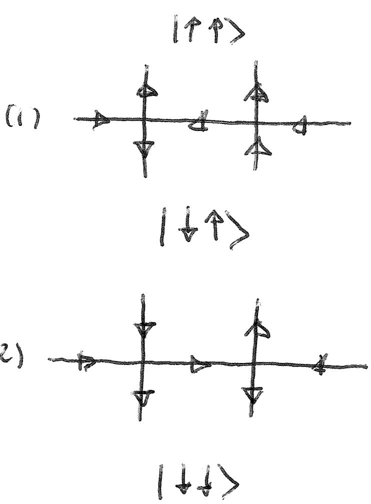
interpretation

$\mathcal{N}_+(\lambda_1, \underline{\mu})|\uparrow\uparrow\rangle$ " = all admissible configurations of line (1) ",

$\mathcal{N}_+(\lambda_2, \underline{\mu})\mathcal{N}_+(\lambda_1, \underline{\mu})|\uparrow\uparrow\rangle$ " = all admissible configurations of line (2) ".

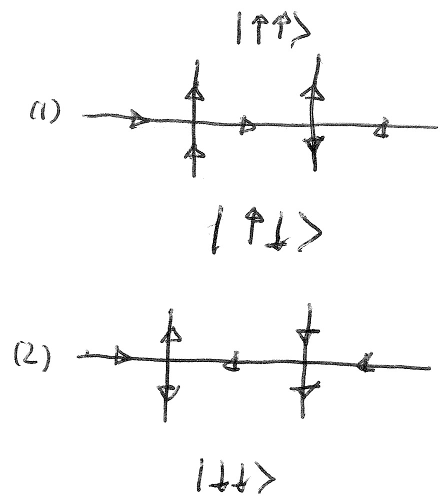
So the inductive process goes as follows:



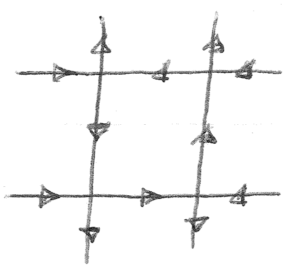
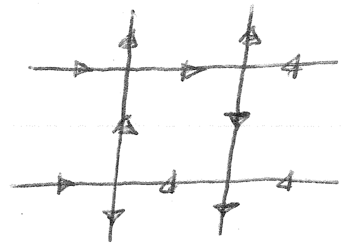


applying $\mathcal{N}_+$

applying $\mathcal{N}_+$ again



This gives the two admissible configurations with their corresponding weight:

 , 

$(q-q^{-1})^2 f(\lambda_1/\mu_2)\, g(\lambda_2/\mu_1)$ $\qquad\qquad$ $(q-q^{-1})^2 g(\lambda_1/\mu_1)\, g(\lambda_2/\mu_2)$.

From the analysis of the algebraic Bethe ansatz and the interpretation of it as generating the lattice configurations of the six vertex model we define the partition function of the six vertex configurations of size N to be the function $Z_N(\underline{\lambda},\underline{\mu})$ defined by:

$$\prod_{m=1}^{N} \mathcal{N}_+(\lambda_m,\underline{\mu})\,|\uparrow\cdots\uparrow\rangle = Z_N(\underline{\lambda},\underline{\mu})\,|\downarrow\cdots\downarrow\rangle .$$

It is the sum over all the six vertex configurations of size N of the product of the weights associated to the vertex by the correspondence:

$$Z_N(\underline{\lambda},\underline{\mu}) = \sum_{C\in SV_N} \prod_{v\in \mathrm{Vertices}(C)} W(v)$$

where $W(v)$ is the weight of the vertex v in configuration C.

In the limit where all the weights are equal to one then $Z_N(\underline{\lambda},\underline{\mu}) \longrightarrow \# SV_N = \# ASM_N$ which solves our enumeration problem.

Thanks to the definition of the partition function it is possible to evaluate $\# ASM_N$ by finding an explicit expression of $Z_N(\underline{\lambda},\underline{\mu})$.

<u>Lemma</u>: The partition function $Z_N(\underline{\lambda},\underline{\mu})$ is a symmetric function of $\lambda_1,\dots,\lambda_N$ and $\mu_1,\dots,\mu_N$ independently.

<u>Proof</u>: For $\lambda_1,\dots,\lambda_N$ it follows from the fact that

$$\mathcal{N}_+(\lambda_i,\underline{\mu})\,\mathcal{N}_+(\lambda_j,\underline{\mu}) = \mathcal{N}_+(\lambda_j,\underline{\mu})\,\mathcal{N}_+(\lambda_i,\underline{\mu}).$$

For $\mu_1,\dots,\mu_N$ we remark that the partition function can be rewritten as:

$$Z_N(\underline{\lambda},\underline{\mu}) = \langle \downarrow\cdots\downarrow| \otimes \langle\uparrow\cdots\uparrow|\, L_{a_N,N}(\lambda_N/\mu_N)\cdots L_{a_N,1}(\lambda_N/\mu_1)\cdots$$

$$\cdots L_{a_1,N}(\lambda_1/\mu_N)\cdots L_{a_1,1}(\lambda_1/\mu_1)\,|\downarrow\cdots\downarrow\rangle \otimes |\uparrow\cdots\uparrow\rangle$$

$$= \langle \downarrow\cdots\downarrow| \otimes \langle\uparrow\cdots\uparrow|\, L_{a_N,N}(\lambda_N/\mu_N)\cdots L_{a_1,N}(\lambda_1/\mu_N)\cdots$$

$$\cdots L_{a_N,1}(\lambda_N/\mu_1)\cdots L_{a_1,1}(\lambda_1/\mu_1)\,|\downarrow\cdots\downarrow\rangle \otimes |\uparrow\cdots\uparrow\rangle$$

$$= \langle \downarrow\cdots\downarrow| \prod_{m=1}^{N} \tilde{\mathcal{N}}_+(\underline{\lambda},\mu_m)\,|\uparrow\cdots\uparrow\rangle$$ by the ultralocality equation.

And since we also have that

$$\tilde{\mathcal{N}}_+(\underline{\lambda},\mu_i)\,\tilde{\mathcal{N}}_+(\underline{\lambda},\mu_j) = \tilde{\mathcal{N}}_+(\underline{\lambda},\mu_j)\,\tilde{\mathcal{N}}_+(\underline{\lambda},\mu_i)$$

the partition function is also symmetric in $\mu_1,\dots,\mu_N$.

$\blacksquare$

<u>Lemma</u>: For all $i\in[\![1,N]\!]$, $\lambda_i^{N-1} Z_N(\underline{\lambda},\underline{\mu})$ is a polynomial of degree at most $N-1$ in λ_i^2 and $\mu_i^{N-1} Z_N(\underline{\lambda},\underline{\mu})$ is a polynomial of degree at most $N-1$ in μ_i^2.

<u>Proof</u>: Since the partition function is symmetric in $\{\lambda_i\}$ and $\{\mu_i\}$ it is sufficient to work with λ_1 and μ_1.

The partition function is:

$$Z_N(\underline{\lambda},\underline{\mu}) = \langle\downarrow\cdots\downarrow|\prod_{m=1}^{N}\mathcal{N}_+(\lambda_m,\underline{\mu})|\uparrow\cdots\uparrow\rangle$$

with

$$\mathcal{N}_+(\lambda_1,\underline{\mu}) = \sum_{i_1,\dots,i_{N-1}\in\{1,2\}} [L_{a,N}(\lambda_1/\mu_N)]_{1i_1}\cdots[L_{a,1}(\lambda_1/\mu_1)]_{i_{N-1}2}.$$

In $Z_N(\underline{\lambda},\underline{\mu})$, λ_1 appears only by the action of $\mathcal{N}_+(\lambda_1,\underline{\mu})$. But in any term of the form:

$$[L_{a,N}(\lambda_1/\mu_N)]_{1i_1}\cdots[L_{a,1}(\lambda_1/\mu_1)]_{i_{N-1}2}$$

there is at most $N-1$ factors of λ_1 and λ_1^{-1} because going from a coefficient $(1 i_1)$ to $(i_{N-1} 2)$ there is one $q-q^{-1}$ as we have seen in the analysis of the algebraic Bethe ansatz.

So multiplying $Z_N(\underline{\lambda}, \underline{\mu})$ by λ_1^{N-1} eliminates the λ_1^{-1} factors and give at most a polynomial of degree $N-1$ in λ_1^2.

The same argument applies for μ_1 if we write the partition function as:

$$Z_N(\underline{\lambda}, \underline{\mu}) = \langle \downarrow \cdots \downarrow | \prod_{m=1}^{N} \tilde{N}_+(\underline{\lambda}, \mu_m) | \uparrow \cdots \uparrow \rangle .$$

∎

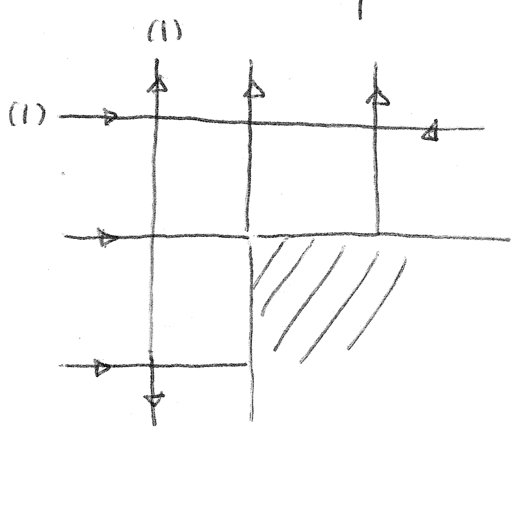
<u>Lemma:</u> The partition function satisfies the following recursion relation:

$$Z_1 = q - q^{-1}$$

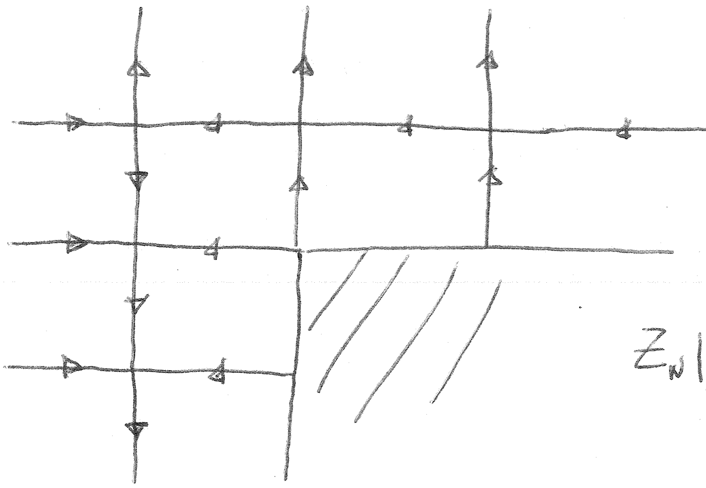
$$Z_N(\underline{\lambda}, \underline{\mu})\big|_{\lambda_1 = \mu_1} = (q-q^{-1}) \prod_{i=2}^{N} \left(\frac{\mu_i}{\lambda_1} q - \frac{\lambda_1}{\mu_i} q^{-1} \right) \left(\frac{\lambda_1}{\lambda_i} q - \frac{\lambda_i}{\lambda_1} q^{-1} \right) Z_{N-1}(\underline{\lambda} \setminus \{\lambda_1\}, \underline{\mu} \setminus \{\mu_1\}).$$

<u>Proof:</u> By definition $Z_1 |\Uparrow\rangle = [L_{a_1,1}(\lambda_1, \mu_1)]_{12} |\uparrow\rangle = q - q^{-1} |\downarrow\rangle$.

The recursion relation is easily obtained by examination of the lattice interpretation:



fixing the (1),(1) vertex to be  completly determine the first line and the first column of the lattice while reproducing the boundary conditions.



Reading the value of the coefficients we get

$Z_N|_{\lambda_1=\mu_1} = (q-q^{-1})$ product of the coefficients on the first line product of the coefficients of the first column Z_{N-1}.

■

<u>Theorem</u>: The partition function of the six vertex configurations of size N is:

$$Z_N(\underline{\lambda},\underline{\mu}) = \frac{\prod_{i,j=1}^{N}\left(\frac{\lambda_j}{\mu_i}-\frac{\mu_i}{\lambda_j}\right)\left(\frac{\lambda_j}{\mu_i}q-\frac{\mu_i}{\lambda_j}q^{-1}\right)}{\prod_{1\le i<j\le N}\left(\frac{\lambda_i}{\lambda_j}-\frac{\lambda_j}{\lambda_i}\right)\left(\frac{\mu_i}{\mu_j}-\frac{\mu_j}{\mu_i}\right)} \det(M)$$

with $$M_{ij} = \frac{q-q^{-1}}{\left(\frac{\lambda_j}{\mu_i}-\frac{\mu_i}{\lambda_j}\right)\left(\frac{\lambda_j}{\mu_i}q-\frac{\mu_i}{\lambda_j}q^{-1}\right)}.$$

<u>Proof</u>: Simply check that this determinental form satisfies the three preceding lemmas.

■