Summary: Cauchy's integral formula and its consequences.

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There are several results for functions in a single complex variable which generalise naturally to $\mathbb{C}^n$. We look at Cauchy's integral formula and some of its consequences in several complex variables.

For many of the proofs, we work over a region called a polydisc, which is a generalisation of the disc. Let $\rho = (\rho_1, ..., \rho_n)$. Then the *open n-polydisc* centred at z_0 and with radius vector r is defined to be

$$_n\mathbb{D}(z_0, \rho) = \{z \in \mathbb{C}^n : |z_i - z_{0,i}| < \rho_i\}.$$

The polydisc has two boundaries which we might need. The first is the *natural boundary*, defined as

$$\partial_n\mathbb{D}(z_0, \rho) = \bigcup_{i=1}^{n} {}_n\Gamma_i(z_0, \rho),$$

where

$$_n\Gamma_i(z_0, \rho) = \{z \in \mathbb{C}^n : |z_i - z_{0,i}| = \rho_i, |z_j - z_{0,j}| \leq \rho_j \ j \neq i\}.$$

The second is the skeleton

$$_n\Gamma(z_0, \rho) = \bigcap_{i=1}^{n} {}_n\Gamma_i(z_0, \rho).$$

Theorem 1. *If f is a function continuous in a neighbourhood U of the closed polydisc $_n\bar{\mathbb{D}}(z_0, \rho)$ and holomorphic in each variable z_i at each point of U, then*

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - z_n| = \rho_n} \cdots \int_{|\zeta_1 - z_1| = \rho_1} \frac{f(\zeta_1, ..., \zeta_n) d\zeta_1 ... d\zeta_n}{(\zeta_1 - z_1)...(\zeta_n - z_n)},$$

for any $z \in {}_n\mathbb{D}(z_0, \rho)$.

Proof. Fix the first $n-1$ variables and and apply the single variable CIF to the resulting function of z_n. Repeat inductively, working from z_n to z_1 to get the result. $\square$

Observation 1. Observe that we could choose to start with any variable, and work through them as we wish. This implies that we can apply Fubini's theorem, allowing us to commute the order of integration in the iterated integral. The question to ask here is whether our hypotheses are strong enough to allow the application of Fubini's theorem.

As long as f is measurable on each circle (it's continuous in a neighbourhood containing each circle, and continuous functions are Borel functions) and each circle is a complete measure space (which it is - consider it as a homeomorphic map of a line segment) then we can apply Fubini's theorem.

Observation 2. This is the most obvious generalisation of CIF, but for singularity analysis it's not great. We will look into a better generalisation towards the end of the seminar series.

We now turn to some results which are consequences of CIF. First is Osgood's lemma. For an open set U, we take $\mathcal{H}(U)$ to be the set of functions holomorphic on U.

Lemma 1. *(Osgood's Lemma) Let $U \subseteq \mathbb{C}^n$ be an open set and $f \in \mathcal{H}(U)$. Then for any polydisc $_n\mathbb{D}(z_0, \rho) \subseteq U$ there is a power series*

$$\hat{f}_{z_0} = \sum_{\alpha_1 + ... + \alpha_n \geq 0} c_{\alpha_1, ..., \alpha_n} (z_1 - z_{0,1})^{\alpha_1} ... (z_n - z_{0,n})^{\alpha_n}$$

that converges uniformly to f on any compact set in $_n\mathbb{D}(z_0, \rho)$.

Proof. Fix z_0 in U and choose two radius vectors $\rho, r \in \mathbb{R}^n_+$ so that we have a nesting of compact polydiscs ${}_n\bar{\mathbb{D}}(z_0,\rho) \subset {}_n\bar{\mathbb{D}}(z_0,r) \subset U$.

We make the substitution of

$$\frac{1}{(\zeta_1 - z_1)...(\zeta_n - z_n)} = \sum_{\alpha_1+...+\alpha_n \geq 0} \frac{(z_1 - z_{0,1})^{\alpha_1}...(z_n - z_{0,n})^{\alpha_n}}{(\zeta_1 - z_{0,1})^{\alpha_1+1}...(\zeta_n - z_n)^{\alpha_n+1}},$$

into the CIF. This comes from the Taylor series expansion of the rational function on the left. This series converges uniformly in ζ and z for $|z_i - z_{0,i}| \leq \rho_i < r_i = |\zeta_i - z_{0,i}|$, and therefore we can commute the summation and iterated integrals to integrate the series termwise, giving a series in z which is uniformly convergent for $z \in {}_n\bar{\mathbb{D}}(z_0,r)$ with coefficients

$$c_{\alpha_1,...,\alpha_n} = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_n - z_n| = r_n} \cdots \int_{|\zeta_1 - z_1| = r_1} \frac{f(\zeta_1,...,\zeta_n)d\zeta_1...d\zeta_n}{(\zeta_1 - z_1)^{\alpha_1+1}...(\zeta_n - z_n)^{\alpha_n+1}}.$$

□

Lemma 2. *Let $z_0 \in \mathbb{C}^n$ and $f_{z_0} = \sum_{\alpha_1+...+\alpha_n \geq 0} c_{\alpha_1,...,\alpha_n}(z_1 - z_{0,1})^{\alpha_1}...(z_n - z_{0,n})^{\alpha_n}$ be the power series bounded for some radius vector $r \in \mathbb{R}^n_+$. Then f_{z_0} converges on any compact subset of ${}_n\mathbb{D}(z_0, |r - z_0|)$.*

Proof. This is left as an exercise. □

Lemma 3. *Let $U \subset \mathbb{C}^n$ be open and $f \in \mathcal{H}(U)$. Then all partial derivatives of f belong to $\mathcal{H}(U)$.*

Proof. To check this, we must first check that all the partial derivatives of a holomorphic function exist. Since f is holomorphic on U, and therefore holomorphic in each variable separately at every point of U, this is fine. Then we must check the Cauchy Riemann equations in several variables to show that these derivatives are holomorphic. □

Theorem 2. *(Cauchy's Inequality) Let $U \subset \mathbb{C}^n$ be open, $f \in \mathcal{H}(U)$ and f continuous on $\bar{U}$. For any ${}_n\mathbb{D}(z_0,\rho)$, if for all $z \in_n \Gamma(z_0,\rho)$, $|f(z)| \leq R$, then for all $\alpha \in \mathbb{N}^n$ we have*

$$\left| \frac{1}{\alpha_1!...\alpha_n!} \partial^{\alpha_1}_{z_1}...\partial^{\alpha_n}_{z_n} f(z_0) \right| \leq \frac{R}{\rho_1^{\alpha_1}...\rho_n^{\alpha_n}}.$$

Proof. This is proved by looking at the uniformly convergent PS representation given by Osgood's Lemma, which tells us

$$\partial^{\alpha_1}_{z_1}...\partial^{\alpha_n}_{z_n} f(z_0) = \alpha_1!...\alpha_n! c_{\alpha_1,...,\alpha_n}.$$

By rearranging and making the substitution for $c_{\alpha_1,...,\alpha_n}$ given by Osgood's lemma, we obtain an upper bound by taking the integral of the function's maximum modulus on the skeleton ${}_n\Gamma(z_0,\rho)$. This gives the result

$$|c_{\alpha_1,...,\alpha_n}| \leq \frac{1}{(2\pi i)^n}(2\pi i\rho_1)...(2\pi i\rho_n)R\frac{1}{\rho_1^{\alpha_1+1}}\cdots\frac{1}{\rho_n^{\alpha_n+1}} = \frac{R}{\rho_1^{\alpha_1}...\rho_n^{\alpha_n}}.$$

□

Theorem 3. *Let D be a domain in $\mathbb{C}^n$ and $f \in \mathcal{H}(D)$. If there exists $z_0 \in D$ such that f and all partial derivatives of f vanish at z_0, then $f \equiv 0$ on D.*

Proof. By developing f as a power series at a point other than z_0 in D, we can show that the coefficients must equal zero for f and all partial derivatives. □

The last three results, the maximum modulus principle, Liouville's theorem and Schwarz's lemma, are related and so we state them together.

Theorem 4. *(Maximum modulus principle) Let $U \subset \mathbb{C}^n$ be a connected open set. If $f \in \mathcal{H}(U)$ and $|f|$ attains a maximum on U, then f is constant.*

Theorem 5. *(Liouville's theorem) Let $f \in \mathcal{H}(\mathbb{C}^n)$. If f is bounded then f is constant.*

Lemma 4. *(Schwarz lemma) Let $D =_n \mathbb{D}(\vec{0}, \vec{r})$, where $\vec{x} = (x, x, ..., x)$. If $f : D \to \mathbb{C}$ is holomorphic with $f(\vec{0}) = 0$, $|f(z)| \leq M$ for all $z \in D$, then*

$$|f(z)| \leq \frac{||z||}{r^n} M \quad \forall z \in D.$$

We begin by outlining two proofs of the MMP.

Proof. (Maximum modulus principle) The first proof uses the open mapping theorem, which we state:

Let $U \subset \mathbb{C}^n$ be connected. If $f \in \mathcal{H}(U)$, then f maps open sets to open sets.

Suppose that f is not constant and that there exists some $z_0 \in U$ such that $|f(z_0)| = \max_{z \in U} |f(z)|$. We apply the open mapping theorem to an open set V containing z_0. This gives an open set containing $f(z_0)$ with elements of strictly greater modulus, and we arrive at a contradiction.

The second uses the CIF.

Suppose that $M = |f(z_0)| = \max_{z \in U} |f(z)|$. Then a consequence of the CIF is

$$f(z_0) = \frac{1}{Vol(\mathbb{D}(z_0, r))} \int_{\mathbb{D}(z_0, r) \subset U} f(\zeta) dV(\zeta).$$

If $|f(z)| = |f(z_0)|$ in $\mathbb{D}(z_0, r)$ then f is constant on D and can be analytically continued to be constant on U. So suppose that f is not constant on $\mathbb{D}(z_0, r)$. Then there is some disc $\mathbb{D}(z_1, r_1) \subset \mathbb{D}(z_0, r)$ on which $f(z) < M - \epsilon$. Thus

$$\begin{aligned} M \cdot Vol(\mathbb{D}(z_0, r)) &= \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_1, r)} f(\zeta) dV(\zeta) + \int_{\mathbb{D}(z_1, r_1)} f(\zeta) dV(\zeta) \\ &\leq M(Vol(\mathbb{D}(z_0, r)) - Vol(\mathbb{D}(z_1, r_1))) + (M - \epsilon) Vol(\mathbb{D}(z_1, r_1)), \end{aligned}$$

arriving at a contradiction. □

Exercise: If $|f|$ constant on U, then f constant.

Proof. (Liouville's theorem) Let f be a bounded function holomorphic on $\mathbb{C}^n$. Then

$$|f(z) - f(z_0)| = |z||f_{z_0} - f(z_0)| = |z||P|,$$

is bounded, where f_{z_0} is the power series for f developed around z_0 and $f(z_0)$ is the first term in the series, thus allowing the factor of $|z|$ on the right. By taking $|z| \to \infty$ any way, we find that $|P| \to 0$, since f bounded implies that $|f(z) - f(z_0)|$ is bounded.

Thus there exists some radius vector ρ such that on $\mathbb{D}(z_0, \rho)$, $|P|$ attains its max, so by the MMP, P is constant on $\mathbb{C}^n$, giving $P = 0$ and $f(z) = f(z_0)$ for all z. □

Proof. (Schwarz lemma) Define

$$g(z) = \frac{f(z)}{z}.$$

This is legitimate, since $f(\vec{0}) = 0$, implying a zero constant term in the PS representation of f. Thus $g(0)$ is defined and g is holomorphic on D.

Take $z \in D$. Then there is some radius ρ such that $r > \rho > 0$ and $z \in \mathbb{D}(\vec{0}, \vec{\rho})$. By the MMP, g will only obtain a maximum on the boundary of $\mathbb{D}(\vec{0}, \vec{\rho})$. Thus, there exists a $z_\rho \in \partial \mathbb{D}(\vec{0}, \vec{\rho})$ such that

$$|g(z)| = \left| \frac{f(z)}{z} \right| \leq \left| \frac{f(z_\rho)}{z_\rho} \right| \leq \frac{M}{r^n}$$

giving the result. □