SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm

Math 303 Fall 2011

Instructor: Dr. Yeats

November 3, 2011

Name: Solutions (please print)

family name *given name*

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Instructions:

(1) Do not open this booklet until told to do so.
(2) Write your name above. Write your email ID on the line provided for it.
(3) Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
(4) Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
(5) This exam has 9 questions on 8 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
(6) **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination.
(7) **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question	Maximum	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	5	
7	5	
8	5	
9	3 BONUS	
Total	25	

(1) **(2 points)** $\boxed{\text{Do \textbf{one} of the following three questions.}}$

(a) Let $\mathcal{C} = \{\{a, \{b\}, d\}, \{b, e\}, \{\{a\}\}\}$. Calculate $\bigcup \mathcal{C}$.

(b) Let $E = \{\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}\}$ and $A = \{\{\{\emptyset\}\}\}$. Calculate A'.

(c) Let $E = \{a, b, \{c\}\}$. Calculate $\mathcal{P}(E)$.

a) $\bigcup \mathcal{C} = \{a, \{b\}, d\} \cup \{b, e\} \cup \{\{a\}\}$
$= \{a, \{b\}, d, b, e, \{a\}\}$

b) $A' = E \setminus A = \{\{\phi\}, \{\{\phi\}\}, \{\{\{\phi\}\}\}\} - \{\{\{\phi\}\}\}$
$= \{\{\phi\}, \{\{\{\phi\}\}\}\}$

c) $\mathcal{P}(E) = \{\phi, \{a\}, \{b\}, \{\{c\}\}, \{a,b\}, \{a, \{c\}\}, \{b, \{c\}\},$
$\{a, b, \{c\}\}\}$

(2) **(2 points)** $\boxed{\text{Do \textbf{one} of the following two questions.}}$

(a) Let $A = \{1, 2\}$ and $B = \emptyset$. Give an example of a set which

(i) contains A as a subset and B as an element

(ii) contains B as a subset and A as an element

(b) Let A be any set, show that $A \cap \emptyset = \emptyset$.

a) (i) let $E = \{1, 2, \phi\}$ then $A \subseteq E$ and $B \in E$

(ii) let $E = \{\{1, 2\}\}$ then $A \in E$ and $B \subseteq E$

b) Take any $x \in A \cap \phi$. Then $x \in A$ and $x \in \phi$
but there are no $x \in \phi$ so there are no $x \in A \cap \phi$
Thus $A \cap \phi = \phi$

(3) **(2 points)** Do **one** of the following two questions.

(a) Write 4 using only $\{$, $\}$, and $\emptyset$.

(b) Let $f : 3 \to \omega$ be defined by $f(n) = n^2$ for $n \in 3$. Write f as a set; you may use ordered pairs and natural numbers without expanding them into sets.

a) $4 = \{0, 1, 2, 3\}$

$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

b) $f = \{(0, 0), (1, 1), (2, 4)\}$

(4) **(2 points)** Do **one** of the following two questions.

(a) (i) Is $\{\{\{1, 2, 3\}\}\}$ an ordered pair? If so what are the first and second coordinates?

(ii) Is $\{\{a, \{\{b\}\}\}, \{a\}\}$ an ordered pair? If so what are the first and second coordinates?

(b) Is $\{(a, a), (b, d), (a, c), (a, d), (b, c)\}$ a cartesian product? If so what sets is it the cartesian product of?

a) (i) Yes $\{\{\{1, 2, 3\}\}\}$ is $(\{1, 2, 3\}, \{1, 2, 3\})$

so both coordinates are $\{1, 2, 3\}$

(ii) Yes $\{\{a, \{\{b\}\}\}, \{c\}\}$ is $(a, \{\{b\}\})$

so the first coordinate is a and the second coordinate is $\{\{b\}\}$

(5) **(2 points)** $\boxed{\text{Do } \textbf{one} \text{ of the following three questions.}}$

(a) Express $x \subseteq y$ in our formal language without using any abbreviations.

(b) Which of the following are well formed formulas (don't worry about parentheses as long as the meaning is clear).

(i) $\forall x((x = c) \wedge (\sim x))$

(ii) $\forall x \exists x((y \in x) \vee (x \in z))$

(iii) $x \in x$

(c) Show that $(A \leftrightarrow B) \vee (B \leftrightarrow \sim A)$ is identically true using a truth table.

a) $x \subseteq y$ is $\forall z\,(z \in x \to z \in y)$

b) (i) not well formed

(ii) well formed

(iii) well formed

c)

A	B	$A \leftrightarrow B$	$\sim A$	$B \leftrightarrow \sim A$	whole thing
F	F	T	T	F	T
F	T	F	T	T	T
T	F	F	F	T	T
T	T	T	F	F	T

these are all trues so the formula is identically true.

(6) **(5 points)** $\boxed{\text{Do } \textbf{one} \text{ of the following three questions.}}$

(a) Let A, B, and C be any sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(b) Let E be any set and let $A, B \subseteq E$. Show that $A \subseteq B$ if and only if $B' \subseteq A'$.

(c) Is the function $f : \omega \to \omega$ defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

one-to-one? Onto? Justify your answers.

a) Take $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$

that is, $x \in A$ and $x \in B$ or $x \in C$

that is, ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

that is, $x \in A \cap B$ or $x \in A \cap C$

that is $x \in (A \cap B) \cup (A \cap C)$

so $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Similarly take $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap B$ or $x \in A \cap C$

that is, ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

that is, $x \in A$ and ($x \in B$ or $x \in C$)

that is, $x \in A$ and $x \in B \cup C$

that is, $x \in A \cap (B \cup C)$

so $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Together we get

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Solutions for b) and c) are on p6

(7) **(5 points)** $\boxed{\text{Do } \textbf{one} \text{ of the following three questions.}}$

(a) Let $\mathcal{S}$ be a nonempty set with the property that all elements of $\mathcal{S}$ are successor sets. Show that $\bigcap \mathcal{S}$ is a successor set.

(b) Show that for any $n \in \omega$, if $x \in n$ then $x \subseteq n$.

(c) Give a one-to-one and onto function between ω and ω^+. Be sure to show your justification that your map is one-to-one and onto.

a) let $T = \cup \mathcal{S}$

$0 \in A$ for all $A \in \mathcal{S}$, therefore $0 \in T$

Take $n \in T$. Then $n \in A$ for all $A \in \mathcal{S}$

so $n^+ \in A$ for all $A \in \mathcal{S}$

Therefore $n^+ \in T$

Thus T is a successor set

b) Use induction. Let S be the set of natural numbers n for which it is true that if $x \in n$ then $x \subseteq n$

Note that there are no $x \in 0$ so trivially $0 \in S$

Take $n \in S$

Consider $n^+ = n \cup \{n\}$

Take $x \in n^+$. There are two possibilities: either $x = n$ or $x \in n$

If $x = n$ then immediately $x \subseteq n^+$

If $x \in n$ then as $n \in S$ $x \subseteq n$ so $x \subseteq n^+$

In both cases n^+ satisfies the desired property so $n^+ \in S$

Therefore by the principle of mathematical induction $S = \omega$

and so we conclude that for any $n \in \omega$

$x \in n \Rightarrow x \subseteq n$

for part c) see page 7

(8) **(5 points)** Do **one** of the following three questions.

(a) Explain how the axiom of subset selection lets us get around Russell's paradox

(b) Explain how the axiom of choice can be viewed both as completely natural and as unreasonable. Be sure to discuss both sides of the issue.

(c) Give a formulation or interpretation of the liar's paradox different from the ones discussed in class.

Solutions will vary

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6 b) Suppose $A \subseteq B$. Take any $x \in B'$

Then $x \in E$ and $x \notin B$

but $A \subseteq B$ so $x \notin B \Rightarrow x \notin A$

Thus $x \in E$ and $x \notin A$

so $x \in A'$. Therefore $B' \subseteq A'$

Suppose $B' \subseteq A'$. Take any $x \in A$. Note $A = (A')'$

So $x \in E$ and $x \notin A'$

but $B' \subseteq A'$ so $x \notin A' \Rightarrow x \notin B'$

Thus $x \in E$ and $x \notin B'$

so $x \in (B')' = B$ Therefore $A \subseteq B$

c) f is not one-to-one as

$f(2) = 1$ and $f(3) = \frac{3-1}{2} = 1$

so $f(2) = f(3)$ but $2 \neq 3$.

f is onto as for any $n \in \omega$

$f(2n) = n$

~~More space for the preceding question, if useful.~~

7 c) Define $f: \omega^+ \to \omega$

by $f(n) = n^+$ for $n \in \omega$

$f(\omega) = 0$

$\omega^+ = \omega \cup \{\omega\}$ so this defines a function and by construction it maps into ω

one-to-one: Say $f(a) = f(b)$

if $f(a) = f(b) = 0$ then since no natural number has 0 as a successor we get $a = b = \omega$

otherwise $f(a) = f(b) = n^+$ for some $n \in \omega$ so $a = b = n$.

onto: Take $n \in \omega$

if $n = 0$ then $f(\omega) = n$

if $n \neq 0$ then $n = m^+$ for some $m \in \omega$ (namely $m = n-1$)

so $f(m) = n$.

(9) **(BONUS 3 points (your max score for the exam is 25))** Give an outline of the content of the course so far. Try to make it tell a coherent story.

just a list was worth 2 points

to get 3 you needed to impose some structure (more than chronological order) on the topics — this is how they tell a story.