Math 303, Fall 2011, Lecture 12

① Free and bound variables

Are the following well formed? (don't worry about parentheses as long as it's clear)

$$x = y$$

$$\forall x \exists y \exists z (y = z)$$

$$\exists x (y \in z)$$

$$\exists x \forall x (x \in y)$$

$$\forall y (\exists x (x \in y) \wedge \exists x (y \in x))$$

to describe what is unpleasant about these we need a definition

Definition Each occurence of a variable symbol in a wff is free or bound as described below

(A) Every variable appearing in

(B) An occurence of a variable in a formula coming from rule (3) is free or bound according to

(

(C)

eg $x \in y$ free

eg $\forall x\,(x \in y)$ bound

eg $(x \in y) \wedge (\forall x\,(x \in y))$

eg $\exists x\,((x \in y) \wedge (\forall x\,(x \in y)))$

eg $\forall y\,\exists x\,((x \in y) \wedge (\forall x\,(x \in y)))$

But there is something very unpleasant going on even though these are well formed

Definition A wff is **good** if every time (3) was applied and every time (4) was applied

Note

The only tricky question is what do

$\forall x\, A$

$\exists x\, A$ mean if x doesn't appear in A?

Answer

eg Which of the formulas given so far today are good

If you stick to good formulas

Definition A formula

a sentence or statement

eg $\forall x \exists y (x \in y)$

eg $\forall x \exists y (y \in x)$

eg $\exists y (x \in y)$

Note a sentence is something which can be true or false

Try the above examples

But be careful

② Some important abbreviations

$x \subseteq y$ abbreviates

$x = \{y\}$ abbreviates

You try the next 3 for the break

$x = \{y, z\}$ abbreviates

$x = \bigcup y$ abbreviates

$x = (y, z)$ abbreviates

We could have used a more consise language if we had wanted to

eg instead of

$A \leftrightarrow B$ we could write

eg instead of

$A \to B$ we could write

Can we be more consise?

③ Propositional calculus

Definition A *propositional function* on the letters $A_1, \dots, A_n$ is a string of symbols defined as follows:

①

② If P and Q are propositional functions then so are

Idea

Note

We think of these as functions

eg $n=3$

eg $A_1 \wedge (A_2 \vee A_3)$

As functions they are defined by the following truth tables

$A_1 \downarrow$	
F	
T	

$\neg A_1$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F		
T		

$A_1 \wedge A_2$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F		
T		

$A_1 \vee A_2$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F		
T		

$A_1 \rightarrow A_2$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F		
T		

$A_1 \leftrightarrow A_2$

A_1	A_2	$A_1 \rightarrow A_2$	$\neg A_1$	$(\neg A_1) \vee A_2$

as an eg lets check

$A_1 \rightarrow A_2$ is the same as $(\neg A_1) \vee A_2$

eg Use a truth table to evaluate

$$(\neg(A \wedge B)) \vee (A \vee B)$$

Is this a $\overbrace{\text{conjunction}}$ you recognize

A	B	$A \vee B$	$A \wedge B$	$\neg(A \wedge B)$	the whole thing

eg Show $A \wedge (\neg A)$ is always true using a truth table

A	$\neg A$	$A \wedge (\neg A)$
F		
T		

If a propositional function is always true we say it is identically true.

We want rules to deduce valid statements. These should match our intuitive notion of true

Rule A Let P be a propositional function in the letters $A_1, A_2, \dots, A_n$. If P is identically true then P with each A_i replaced by any sentence is a valid statement

Rule B If A and $A \to B$ are valid statements then so is B

These together give us

Reductio ad absurdum (negation introduction)

From p and [accepting q leads to a proof that $\neg p$], infer $\neg q$.

Double negative elimination

From $\neg\neg p$, infer p.

Conjunction introduction

From p and q, infer $(p \wedge q)$.

From p and q, infer $(q \wedge p)$.

Conjunction elimination

From $(p \wedge q)$, infer p.

From $(p \wedge q)$, infer q.

Disjunction introduction

From p, infer $(p \vee q)$.

From p, infer $(q \vee p)$.

Disjunction elimination

From $(p \vee q)$ and $(p \rightarrow r)$ and $(q \rightarrow r)$, infer r.

Biconditional introduction

From $(p \rightarrow q)$ and $(q \rightarrow p)$, infer $(p \leftrightarrow q)$.

Biconditional elimination

From $(p \leftrightarrow q)$, infer $(p \rightarrow q)$.

From $(p \leftrightarrow q)$, infer $(q \rightarrow p)$.

Modus ponens (conditional elimination)

From p and $(p \rightarrow q)$, infer q.

Conditional proof (conditional introduction)

From [accepting p allows a proof of q], infer $(p \rightarrow q)$.

(from Wikipedia "propositional calculus")

These are the usual deduction rules for propositional calculus.

We get then as, for example,

You can check that everything above comes from an identically true implication

Tougher is showing the 10 rules above are enough to derive all identically true propositional functions.

4 Next time

The liar's paradox