# Math 303, Fall 2011, Lecture 2

## ① The empty set

So far we don't have any specific sets from our axioms. Let us start with the simplest one

**Axiom of the empty set** There is a set, written $\emptyset$, which contains no elements

Note ① There is only one empty set

The axiom of extension tells us that a set is determined by its elements, so in this case if two sets both have no elements they must be the same set.

② The empty set is a subset of every set.

Recall $A$, $B$ sets then $A \subseteq B$ means

(every $x \in A$ is also in $B$)

this is vacuous for $A = \emptyset$

$\emptyset \in \{\emptyset, 1, 2, 3\}$

$\emptyset \subseteq$

$\phi \notin \phi$ $\qquad\qquad$ $A \cup \phi = A$

Tip: sometimes when working with $\phi$ or other vacuous things its easier to turn it around

suppose $\phi \not\subseteq A$ then there must be some $x \in \phi$ but $x \notin A$

but there is nothing in $\phi$ so in particular no $x \in \phi$ with $x \notin A$.

## ② Building sets part 1 : pairs

We started by writing out sets explicitly eg $\{1,2,3\}$
but we don't yet have axioms to do that.

Here is a start things like $\{1,2\}$

**Axiom of pairing (or unordered pairs)**

For any two sets $A$ and $B$, there is a set $C$ with $A \in C$ and $B \in C$ and $\boxed{\text{nothing else}}$

write this as $C = \{A, B\}$

Halmos doesn't require this

if I have $D$ with $A \in D$ and $B \in D$ and other stuff. How do I get $\{A,B\}$.

Use axiom of subsets selection (specification)

$$\{A,B\} = \{x \in D \mid x = A \text{ or } x = B\}$$

eg $A = B = \emptyset$

$$\{A,B\} = \{\emptyset\}$$

eg $A = \{\emptyset\}$ $B = \emptyset$

$$\{A,B\} = \{\emptyset, \{\emptyset\}\}$$

eg $A = \{\emptyset, \{\emptyset\}\}$, $B = \{\{\emptyset\}\}$

$$\{A,B\} = \{\{\emptyset,\{\emptyset\}\}, \{\{\emptyset\}\}\}$$

eg
$A=1$
$B=1$
$\{A,B\}$
$=\{1\}$.

Note ① $\{A,B\} = \{B,A\}$ (hence unordered pair)

② $\{A,A\} = \{A\}$

lets check that $\phi, \{\phi\}, \{\{\phi\}\}, \{\{\{\phi\}\}\}, \ldots$ are all distinct.

Suppose not ~~Then~~

$$\underbrace{\{\ldots\{}_{i}\phi\}\ldots\} = \underbrace{\{\ldots\{}_{j}\phi\}\ldots\}$$

wlog $\underline{\underline{0<i}}<j$ so write

$$\underbrace{\{\ldots\{}_{i}\phi\}\ldots\} = \underbrace{\{\ldots\{}_{i}\underbrace{\{\ldots\{}_{j-i}\phi\}\ldots\}\}\ldots\}$$

by the axiom of extension a set is determined by its elements

in this case each (outer) set has only 1 element

formally induction

iterated $i-1$ times

so $\underbrace{\{\ldots\{}_{i-1}\phi\}\ldots\} = \underbrace{\{\ldots\{}_{i-1}\underbrace{\{\ldots\{}_{j-i}\phi\}\ldots\}\}\ldots\}$

continue until we get

$$\phi = \underbrace{\{\ldots\{}_{j-i}\phi\}\ldots\}$$

this is a contradiction since $\emptyset$ has no elements but $\{\underbrace{\ldots}_{j-i}\{\emptyset\}\ldots\}$ has 1 element

finally if $i=0$, $j>i$ we have $\emptyset = \{\underbrace{\ldots}_{j}\{\emptyset\}\ldots\}$

also a contradiction.

http://math.sfu.ca/~kyeats/teaching/math303.html

Use the same ideas to show that all sets made from $\emptyset$ and the pairing axiom are distinct.

## ② Building sets part 2: unions

other natural thing to do is collect together the elements of two sets into a new set. This is union

Let $A$ and $B$ be sets. Then there is a set $A \cup B$ which contains the elements of $A$ and the elements of $B$ and nothing else

as before

Note ① $A \subseteq A \cup B$ $\qquad$ $B \subseteq A \cup B$

② Compare this to the pairing axiom

$\{1\} \cup \{2\} = \{1, 2\}$ which is the pair of 1 and 2

but the pair of $\{1\}$, $\{2\}$ get $\{\{1\}, \{2\}\}$.

(3) Now we can make sets with more than two elements

$\emptyset$

pairing: $\{\emptyset, \emptyset\} = \{\emptyset\}$

pairing: $\{\emptyset, \{\emptyset\}\}$

pairing $\{\{\emptyset\}, \{\emptyset\}\} = \{\{\emptyset\}\}$

pairing $\{\{\emptyset\}, \{\{\emptyset\}\}\}$

$$\{\emptyset\} \cup \{\{\emptyset\}, \{\{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$$

But we want to do more than this

rather than just taking the union of 2 sets

how about more. lets allow unions of any set of sets

## Axiom of Unions

Let $\mathcal{C}$ be a set of sets. Then there is a set which contains all elements which belong to at least one set from $\mathcal{C}$, and nothing else

eg let $\mathcal{C} = \{\{1\}, \{1,2,4\}, \{\{986\},8\}, \{\emptyset\}\}$

then $\bigcup \mathcal{C} = \{1,2,4,\{986\},8,\emptyset\}$

Note ① In set theory the usual notation for this

$\bigcup \mathcal{C}$

In the rest of math more usual $\bigcup_{A \in \mathcal{C}} A$

② Another way to say it is that $\bigcup \mathcal{C}$ is the set containing all elements of elements of $\mathcal{C}$

eg let $A_n = \{1, 2, \dots, n\}$ for any positive $n$

let $\mathcal{C} = \{A_1, A_2, \dots\}$

what is $\bigcup \mathcal{C} = \bigcup_{n=1}^{\infty} A_n = \{1, 2, 3, \dots\}$

eg let $A_n = \{n^2, n^3, n^4, \dots\}$ let $\mathcal{C} = \{A_1, A_2, \dots\}$

what is $\bigcup \mathcal{C} = \{1, 4, 8, 9, 16, 25, 27, \dots\} = \{n^k \mid n \geq 1, k \geq 2, k \in \mathbb{Z}\}$

Properties and special cases of unions

① What is $\bigcup \emptyset = \emptyset$

② What is $\bigcup \{A\} = A$

③ What is $\bigcup \{A, B\} = A \cup B$

4 What is $A \cup \emptyset$ ? $A$

5 What is $A \cup A$ ? $A$

Also Note 1 $A \cup B = B \cup A$

2 $(A \cup B) \cup C = A \cup (B \cup C) = \bigcup \{A, B, C\}$.

3 Next time

Power sets, and things we can construct with our axioms so far

$\boxed{\text{Please read sections 5 and 6 of Halmos}}$