Math 303, Fall 2011, Lecture 20

① Properties of ordinals and well orders

$\boxed{\text{Every element of an ordinal number } X \text{ is also a subset of } X \text{ ie if } X \text{ is an ordinal and } x \in X \text{ then } x \subseteq X}$

we called such sets transitive.

proof let X be an ordinal and take $x \in X$

then $x = s(x)$

but $s(x) = \{ y \in X \mid y < x \}$

in particular $s(x) \subseteq X$

so $x \subseteq X$

Note this is a property we already saw for natural numbers

Next lets define two well ordered sets X and Y to be similar if there is a function $f: X \to Y$ which is

- one-to-one
- onto
- for all $x_1, x_2 \in X$

$$\underbrace{x_1 \leq x_2}_{\text{order in } X} \text{ in } X \quad \text{if and only if} \quad \underbrace{f(x_1) \leq f(x_2)}_{\text{order in } Y} \text{ in } Y$$

eg let $X = \{2, 8, 9\}$ with $2 \leq 8 \leq 9$

and let $Y = 3 = \{0, 1, 2\}$ with $0 \leq 1 \leq 2$

then X and Y are similar

using
$$f(2) = 0$$
$$f(8) = 1$$
$$f(9) = 2$$

this is one-to-one and onto.

Now check the order property

$2 \leq 8$ and $f(2) = 0 \leq 1 = f(8)$; $8 \leq 9$ and $f(8) = 1 \leq 2 = f(9)$
So X is similar to Y

but not every one-to-one and onto function between X and Y gives a similarity

Say $f(2) = 2$, $f(8) = 0$, $f(9) = 1$ then $f(2) = 2 \not\leq f(8) = 0$
but $2 \leq 8$

eg lets order ω^+ in 2 ways

$X = \omega^+$ with the usual ordering ($n \leq \omega$ for all $n \in \omega$ and natural numbers are ordered as in ω)

$Y = \omega^+$ ordered by

$$\omega \leq 0 \leq 1 \leq 2 \leq \ldots$$

Is X similar to Y?

No because say $f : X \to Y$ was a similarity then $f(\omega) \geq f(n)$ for all $n \in \omega$
so $f(\omega)$ would have to be the largest element of Y. But Y has no largest element, contradiction.

Is Y similar to something we've seen before?

Yes Y is similar to ω

$$f: Y \to \omega$$

$$f(\omega) = 0$$

$$f(n) = n^+ \quad \text{for } n \in \omega.$$

More facts

$\boxed{\text{If two well ordered sets are similar then there is exactly one similarity function between them}}$

proof Let X and Y be the sets.

Say $f: X \to Y$ and $g: X \to Y$ are both similarities

then $g^{-1} \circ f : X \to X$

ie for $x \in X$ this map is $g^{-1}(f(x))$

call this map h.

take $x_1, x_2 \in X$, $x_1 \le x_2$

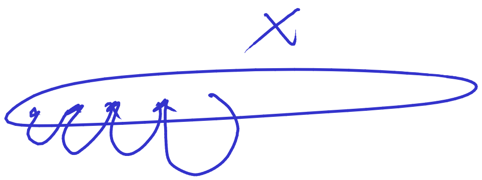
then $f(x_1) \le f(x_2)$

so $g^{-1}(f(x_1)) \le g^{-1}(f(x_2))$

so h also preserves order.

Now consider the set

$$\{x \in X : h(x) < x\} = S$$



X is well ordered so this set has a least element call it a.

so $h(a) < a$

then $h(h(a)) < h(a)$

but this contradicts the minimality of a

(because says $h(a) \in S$

but says $a > h(a)$

so a is not the minimal element of S)

so $h(x) \geq x$ for all $x \in X$

$(g^{-1} \circ f)(x) \geq x$ for all $x \in X$

so $x \geq (f^{-1} \circ g)(x)$

but by the same argument applied to $f^{-1} \circ g$

get $(f^{-1} \circ g)(x) \geq x$ for all $x \in X$

so $(f^{-1} \circ g)(x) = x$ for all x

so $f = g$.

So there is only one similarity between X and Y

A well ordered set cannot be similar to any of its initial segments

proof Let X be a well ordered set and $x \in X$ so that X is similar to $s(x)$

Say with $f: X \to s(x)$

As for the previous fact consider the set

$$S = \boxed{\{y \in X : f(y) < y\}}$$

Note $S \neq \emptyset$ because $f(x) \in s(x)$

so $f(x) < x$

so $x \in S$

let a be the least element of S

so $f(a) < a$ since $a \in S$

so $f(f(a)) < f(a)$

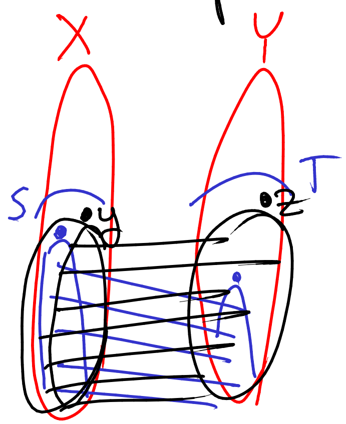
so $f(a) \in S$ but $f(a) < a$ so a wasn't minimal. Contradiction.

So X is not similar to any initial segment of itself

Let X and Y be well ordered sets. Either X and Y are similar or one of them is similar to an initial segment of the other

but not both by the previous fact

proof



Let $S = \{ a \in X : \exists b \in Y \, (s(a) \text{ is similar to } s(b))\}$

Let $T = \{ b \in Y : \exists a \in S \, (s(a) \text{ is similar to } s(b))\}$

Note S and T are similar using the function taking a to b (check)

Now either $S = X$ or $X - S \neq \emptyset$ and so $X - S$ has a least element x.

Claim $\boxed{S = s(x)}$

Since x was the least element not in S every element less than x is in S

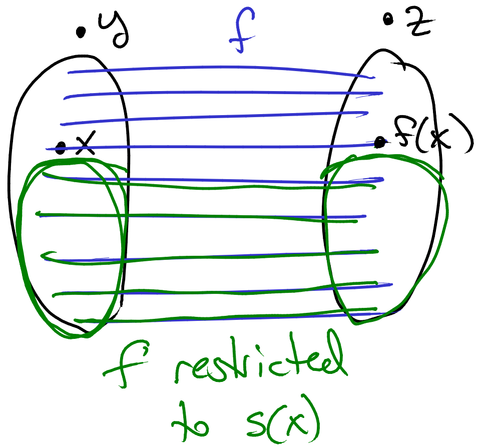
so $s(x) \subseteq S$

What if $s(x) \neq S$. Take $y \in S$ $y > x$

then $s(y)$ is similar to some $s(z)$ for some $z \in Y$

say $f : s(y) \to s(z)$

but $x \in s(y)$



so restrict f to $s(x)$

$f : s(x) \longrightarrow s(z)$

what is the image?

it is the initial segment of $f(x)$

then $s(x)$ is similar to $s(f(x))$ contradicting $x \notin S$.

This gives the claim

Now we can conclude

If $S = X$ and $T = Y$ then X is similar to Y since S is similar to T

If $S = X$ and $T \neq Y$ by the above argument applied to T rather than S get T is an initial segment of Y

so X is similar to an initial segment of Y

If $T = Y$ and $S \neq X$ similarly S is an initial segment of X

so Y is similar to an initial segment of X

If $T \neq Y$ and $S \neq X$

then $T = s(z)$ for some $z \in Y$

$S = s(x)$ for some $x \in X$

but we know S and T are similar

so $s(z)$ and $s(x)$ are similar. So $x \in S$ and $z \in T$

But that contradicts $T = s(z)$ and $S = s(x)$, so this case doesn't occur

eg $6 = \{0, 1, 2, 3, 4, 5\}$

$s(4) = \{0, 1, 2, 3\}$ all the elements of 6 which are strictly less than 4

eg let $X = \{1, 3, 92, 8\}$ with the order

$1 \leq 3 \leq 8 \leq 92$ (this X is not an ordinal)

what is $s(3)$ in X, $s(3) = \{1\}$

Note this is different than $s(3)$ in 6 (say)

in 6, $s(3) = \{0, 1, 2\}$.

② Next time

- Bringing the above back to ordinals
- Sizes of sets

Please read Halmos sections 22 and 23