Math 303, Fall 2011, Lecture 21

① Ordering the ordinals

Last time we saw that two well ordered sets X and Y are similar if there is a function $f: X \to Y$ which is

- one-to-one
- onto
- and for all $x_1, x_2 \in X$

$$x_1 \leq x_2 \quad \text{if and only if} \quad f(x_1) \leq f(x_2)$$

and

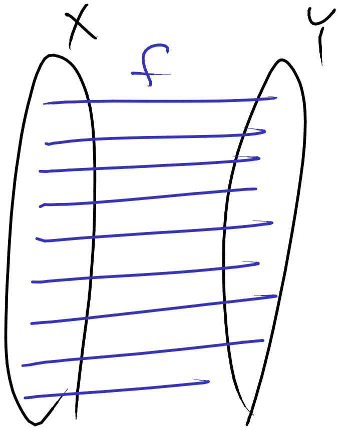
Let X and Y be well ordered sets. Either X and Y are similar or one of them is similar to an initial segment of the other

And not both

For ordinals we get one further similarity property

If two ordinals are similar then they are equal

proof let X and Y be ordinals and $f : X \to Y$ a similarity



let $\boxed{S = \{ x \in X : f(x) = x \}}$

Use transfinite induction (goal show $S = X$)

so we need to check whenever $s(x) \subseteq S$ then $x \in S$

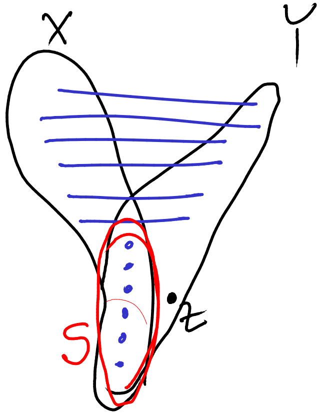
So suppose $x \in X$ and $\boxed{s(x) \subseteq S}$

so $f(x) > f(y)$ for all $y \in s(x)$

but $f(y) = y$ for all $y \in s(x)$

so $f(x) > y$ for all $y \in s(x)$

Furthermore there is no $z \in Y$, $z < f(x)$, $z \notin s(x)$ because if there were then nothing in X could map to it preserving the order



this tells us $s(f(x)) = s(x)$

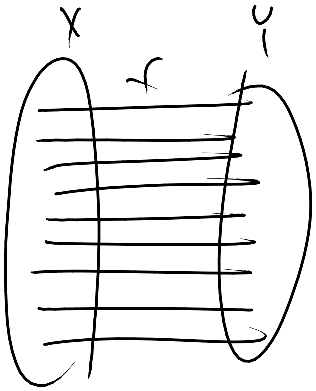
but X and Y are ordinals

so $f(x) = s(f(x)) = s(x) = x$

so $f(x) = x$ so $\boxed{x \in S}$

So by transfinite induction $S = X$

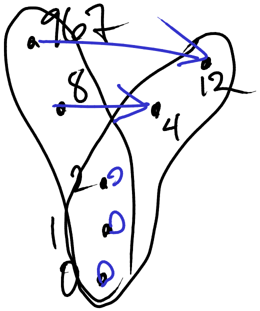
so for all $x \in X$ $f(x) = x$



so $\boxed{Y = X}$

Could have a map that doesn't zipper for well ordered sets that aren't ordinals

eg $X = \{0, 1, 2, 8, 967\}$ $Y = \{0, 1, 2, 4, 12\}$



Putting all this together we get

If X and Y are ordinals then either $X = Y$ or one of them is equal to an initial segment of the other

This uses the previous result

Thus the ordinals themselves are totally ordered.

Say $X < Y$ if X is equal to an initial segment of Y.

eg $2 = \{\boxed{0, 1}\}$ $8 = \{\boxed{\boxed{0, 1}, 2, 3, 4, 5, 6, 7}\}$ so $2 < 8$

$\omega = \{\boxed{\boxed{0, 1}, 2, 3, 4, 5, 6, 7}, 8, \dots\}$ so $2 < \omega$

and $8 < \omega$

In fact this ordering is a well ordering

proof let E be any nonempty set of ordinals.

Take any $X \in E$

$-, -3, -2, -1$

if X is the smallest element of E we're done

Otherwise since E is totally ordered there is a $Y \in E$, $Y < X$

since X is an ordinal $Y \in X$

Note that $Y \in X \cap E$ so in particular $X \cap E \neq \emptyset$

Furthermore $X \cap E \subseteq X$ and X is an ordinal hence X is well ordered.

Take the least element of $X \cap E$ call it Z.

Point Z is also least in E

(check details

take $W \in E$

if $W \geq X$ then $Z \leq X \leq W$

so $Z \leq W$

if $W < X$ then $W \in X$

so $W \in X \cap E$

so $Z \leq W$

In all cases $Z \leq W$

so Z is least in E)

Furthermore the order on ordinals satisfies the ordinal property!!

let X be an ordinal. All ordinals less than X are by definition initial segments of X, and since X is an ordinal all initial segments of X are elements of X so all ordinals less than X are elements of X

Also all elements of X are ordinals and are all $< X$.

Question Is the set of all ordinals an ordinal?

Think about it.

It seems like yes.

For now let $\mathcal{O}$ be set of all ordinals

Consider $\mathcal{O}^+$ We know $\mathcal{O}^+$ is an ordinal since $\mathcal{O}$ is an ordinal

so $\mathcal{O} < \mathcal{O}^+$

but $\mathcal{O}^+ \in \mathcal{O}$ since $\mathcal{O}$ is the set of all ordinals

So $\mathcal{O}^+ < \mathcal{O}$ Contradiction.

Problem $\mathcal{O}$ is not a set.

This is called the Burali-Forti paradox

The Burali-Forti paradox is similar to Russell's paradox but predates it (by a few years)

1897 1901

though all these ideas were in the air at the same time.

($1\frac{1}{2}$) Hw 5 minor avg 7.94

Final exam will be approx half-half material before and after the midterm.

② Counting with ordinals

We have

$0, 1, 2, 3, 4, \ldots$

$\omega, \omega^+, (\omega^+)^+$

write write

$\omega+1, \quad \omega+2, \quad \omega+3, \quad \omega+4, \quad \ldots$

Next is $\{0, 1, 2, 3, \dots, \omega, \omega+1, \omega+2, \dots\}$

$= \omega 2$

← this notation comes from ordinal arithmetic we won't do it but you can read Halmos section 21 if you're interested

order matters in ordinal arithmetic so don't write 2ω

then $\omega 2+1, \omega 2+2, \omega 2+3, \dots$

$\{0, 1, 2, 3, \dots, \omega, \omega+1, \omega+2, \dots, \omega 2, \omega 2+1, \omega 2+2, \dots\}$

$= \omega 3$

$\omega 3+1, \omega 3+2, \omega 3+3 \dots \omega 4, \dots \omega 5, \dots \omega 6 \dots$

$\dots$

ω^2 ← note not exponentiation meaning sets of functions - it is ordinal exponentiation

$\omega^2+1, \omega^2+2, \dots, \omega^2+\omega, \dots \omega^2+\omega 2, \dots$

$\omega^2 2 \dots \omega^2 3 \dots \dots$

$\omega^3 \dots \omega^4 \dots \omega^5 \dots$

ω^{ω} , $\omega^{\omega}+1$ . . . . $\omega^{(\omega+1)}$ . . . .

$\omega^{\omega 2}$ . . . $\omega^{\omega 3}$

. . . . $\omega^{(\omega^{2})}$ . . . . . $\omega^{(\omega^{\omega})}$

. . . . $\omega^{\omega^{\omega^{\omega}}}$ . . . . $\omega^{\omega^{\omega^{\omega^{\omega}}}}$ . . . $\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}}}}$

then put all these things together in a set
this is traditionally called

ε_0

The first one without arithmetic notation.

keeps going ε_0+1 . . . .

For the break is ε_0 countable?

② Size

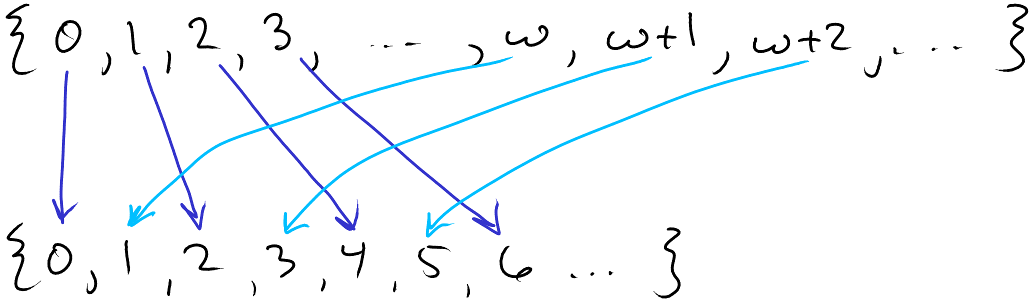
A while ago we said two sets were the same size if there was a one-to-one and onto function between them.

The ordinals don't capture this notion of size

eg ω and ω^+ are the same size (on a homework and on the midterm)

Similarly $\omega+2$, $\omega+3$, … are also the same size as ω

Also $\omega 2$ is the same size as ω



And so on

Two natural questions

① Is there anything bigger than ω in the sense of size

② Are there natural sets which "count" the size.

Recall that we say two sets are equivalent if they are the same size in this sense

We will write $X \sim Y$ to say X is equivalent to Y

(compare to similarity for well ordered sets)

↑ $\boxed{\text{different}}$

Let X and Y be sets. Say Y dominates X and write

$$X \precsim Y$$

a curly $\leq$

if X is equivalent to a subset of Y

$\precsim$

think Y is at least as big (in size) as X

eg $\{0, 2, 4, 6, 8 \dots\} \precsim \omega$ also $X \precsim X$ for all X

We would like $\precsim$ (domination) to give a partial order on sets of sets. However this isn't quite true.

- Suppose $X \precsim Y$ and $Y \precsim Z$. Specifically say

$\left\{ \begin{array}{l} f: X \to Y \text{ which is one-to-one} \\ \text{so by } f, X \text{ is equivalent to } \tilde{Y} \text{, the image of } X \text{ under } f \end{array} \right.$

$\underbrace{f: X \to \tilde{Y}}_{\text{one-to-one and onto}} \leq Y$

Similarly say $g: Y \to \tilde{Z} \subseteq Z$ is one-to-one and onto.

so $g \circ f : X \longrightarrow Z$ is one-to-one since f and g were (check!)

and is onto its image

so X is equivalent to a subset of Z.

Thus $\precsim$ is transitive.

- reflexive is also ok X is equivalent to itself ie $X \precsim X$
- antisymmetry is a problem however.

Suppose $X \precsim Y$ and $Y \precsim X$

but X and Y are not necessarily the same set all we can conclude is

$$X \sim Y$$

This is sufficiently important to be a named theorem

Schröder - Bernstein Theorem

If $X \precsim Y$ and $Y \precsim X$ then $X \sim Y$

This tells us our notion of size is well behaved

The theorem says

if X is smaller (in size) than Y
and Y is smaller (in size) than X
then X and Y are the same size.

I'm not going to prove Schröder - Bernstein because I want to have time to get to some more paradoxes

Another useful fact is

if X and Y are sets then either $X \precsim Y$ or $Y \precsim X$

proof well order X and Y

either X and Y wellordered are similar
in which case forgetting the order we get
$X \sim Y$ so $X \precsim Y$ and $Y \precsim X$

or one of them is similar to an initial segment of the other. Again forgetting the order one of them is equivalent to a subset of the other ie. $X \precsim Y$ or $Y \precsim X$

③ Countability revisited

Write $X \prec Y$ to mean $X \precsim Y$ and $X \not\sim Y$

With this notion of domination, we can say

X is finite if $X \prec \omega$

X is infinite if $\omega \precsim X$

X is countable if $X \precsim \omega$

X is countably infinite if $X \sim \omega$

Note previously we'd said a set was finite if it was equivalent to a natural number. This is the same as saying $X \prec \omega$

Note a set X is countable if $X \precsim \omega$

ie there is an $f : X \to \omega$ which is one-to-one but not necessarily onto.

Some things which are countable

① The union of two countable sets is countable

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \omega & \end{tikzcd}$$

f, g one-to-one

$\omega = \{0, 1, 2, .. \}$

$$\begin{tikzcd} X \arrow[dr, "2f"'] & & Y \arrow[dl, "2g+1"] \\ & \omega = \{0, 1, 2, 3, 4, 5, 6, \dots\} & \end{tikzcd}$$

f, g one-to-one

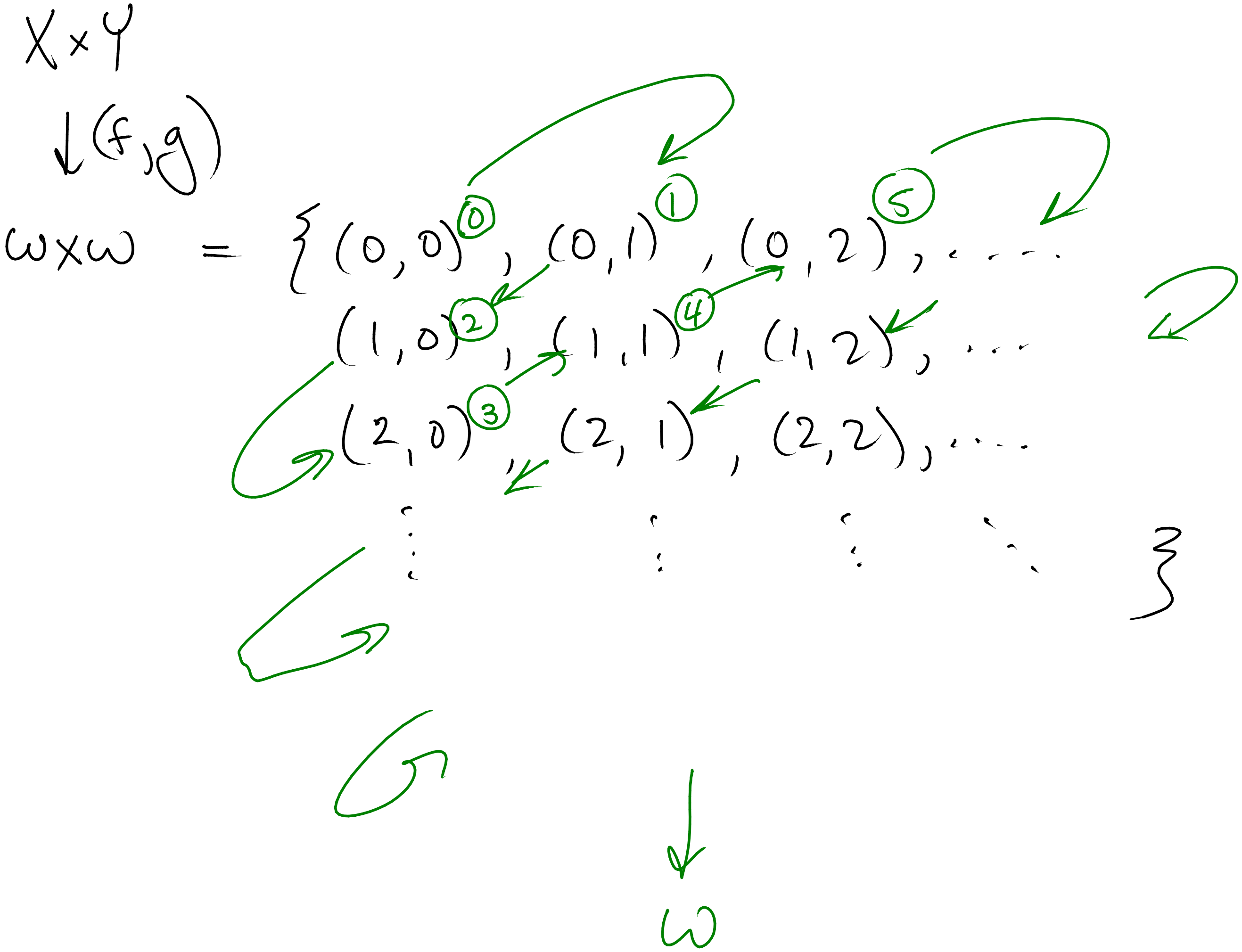
now the ranges are disjoint so put then together

$$\begin{tikzcd} X \cup Y \arrow[d, "2f"', shift right=2] \arrow[d, "2g+1 \text{ for } y \in Y \text{ but } y \notin X", shift left=2] \\ \omega = \{0, 1, 2, 3, 4, 5, 6, \dots\} \end{tikzcd}$$

② The cartesian product of two countable sets is countable

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \omega = \{0, 1, 2, \dots\} & \end{tikzcd}$$

f, g one-to-one



③ The set of finite subsets of a countable set is countable

$$\begin{tikzcd} X \arrow[d, "f"'] \\ \omega = \{0, 1, 2, \ldots\} \end{tikzcd}$$

f one-to-one

$\emptyset$

subsets of X with 1 element whose image under f is ≤ 1

Subsets of X with ≤ 2 elements whose images under f are ≤ 2 (and I haven't alredy listed them)

. . .

$$\omega = \{0, 1, 2, 3, 4, \dots$$

④ Next time

- Cantor diagonalization – not every set is countable
- Cardinal numbers

$\boxed{\text{Please read Halmos section 25}}$